**1.(a)** P{at least one correct} = P{A   B   C} = $1 - P\{A^c \cap B^c \cap C^c\}$ = 1 – P{none correct}
$= 1 - P(A^c)P(B^c)P(C^c) = 1 - 0.2\times0.1\times0.3 = 1 - 0.006 = 0.994$

**(b)** P{only #2 correct} = $P\{A^c \cap B \cap C^c\} = P(A^c)P(B)P(C^c) = 0.2\times0.9\times0.3 = 0.054$

**(c)** P{#3 correct | at least one correct} = P{C | A   B   C} = P{C ∩ [A   B   C]}/P{A   B   C}
= P{C}/P{A   B   C} = 0.7/0.994 = 50/71

**2.(a)** P{**X** > n} = P{mailman not bitten by a dog for the first n days}
= P{not bitten on n-th day **and** not bitten on previous n–1 days}

$= P\{\text{not bitten on n-th} \mid \mathbf{X} > n-1\}P\{\mathbf{X} > n-1\} = [1 - \frac{1}{n+1}] \times P\{\mathbf{X} > n-1\} = \frac{n}{n+1} \times P\{\mathbf{X} > n-1\}$

$= \frac{n}{n+1} \times \frac{n-1}{n} \times P\{\mathbf{X} > n-2\} = \frac{n}{n+1} \times \frac{n-1}{n} \times \frac{n-2}{n-1} \times P\{\mathbf{X} > n-3\} = \frac{n}{n+1} \times \frac{n-1}{n} \times \dots \times \frac{3}{4} \times \frac{2}{3} \times P\{\mathbf{X} > 1\}$

$= \frac{1}{n+1}$ since P{mailman not bitten on first day} $= \frac{1}{2}$ and all the intermediate factors cancel. Hence,

$p_{\mathbf{X}}(n) = P\{\mathbf{X} > n-1\} - P\{\mathbf{X} > n\} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ for n = 1 , 2, 3, …

**(b)** $E[\mathbf{X}] = \sum_{n=1}^{\infty} n \cdot p_{\mathbf{X}}(n) = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ since the harmonic series diverges. Dog lovers everywhere can breathe a sigh of relief! Note that we can also get this using $E[\mathbf{X}] = \sum_{n=1}^{\infty} P\{\mathbf{X} > n\} \dots$

**3.** Let q = 1–p. Then, P{**X** is even} $= pq + pq^3 + pq^5 + \dots = \frac{pq}{1-q^2} = \frac{q}{1+q} = \frac{4}{9}$ which gives $q = \frac{4}{5}$ and $p = \frac{1}{5}$.
Hence E[**X**] = 1/p = 5.

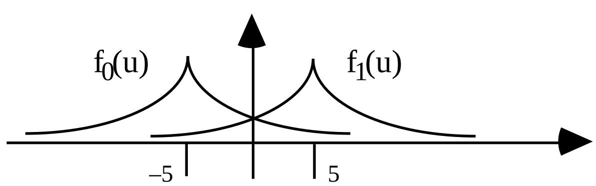
**4.** Which of the following statements are true for **all** random variables **X** and **Y** with identical finite variance $\sigma^2$ ?

| TRUE | FALSE | |
|---|---|---|
| □ | ■ | $E[\mathbf{X}^2] = E[\mathbf{Y}^2]$ |
| □ | ■ | $\text{var}(\mathbf{X} + \mathbf{Y}) = 2\sigma^2$ |
| □ | ■ | $\text{var}(\mathbf{X} - \mathbf{Y}) = 0$ |
| ■ | □ | $\text{var}(\mathbf{X} + \mathbf{Y}) + \text{var}(\mathbf{X} - \mathbf{Y}) = 4\sigma^2$ |
| ■ | □ | $\text{var}(2\mathbf{X} + 3\mathbf{Y}) = \text{var}(3\mathbf{X} + 2\mathbf{Y})$ |
| ■ | □ | ÷cov(**X**, **Y**)÷ ≤ $\sigma^2$ |
| ■ | □ | **X** + **Y** and **X** – **Y** are uncorrelated random variables |
| □ | ■ | **X** + **Y** and **X** – **Y** are independent random variables |

**5.(a)** **Y** = exp(**X**) takes on values in the range (1,∞) as **X** varies between (0,∞).
For v > 1, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{\exp(\mathbf{X}) \le v\} = P\{\mathbf{X} \le \ln v\} = 1 - \exp(-\lambda(\ln v)) = 1 - v^{-\lambda}$. Note λ > 0.

Hence, $f_{\mathbf{Y}}(v) = \begin{cases} \lambda v^{-\lambda-1}, & v > 1, \\ 0, & v \le 1. \end{cases}$

**6.(a)** The pdfs are as shown in the sketch below.




The maximum-likelihood decision chooses the hypothesis which has the larger pdf value at the observation. By inspection, we see that the decision is to choose $H_1$ if **X** > 0 and $H_0$ if **X** < 0.
$P_{FA}$= P{false alarm} = P{$H_1$ is chosen when in fact $H_0$ is the true hypothesis} = P{**X** > 0 when $H_0$ is true}

$= \int_0^{\infty} f_0(u)du = \int_0^{\infty} (1/2)\cdot\exp(-|u+5|)du = \int_0^{\infty} (1/2)\cdot\exp(-u-5)du = (1/2)\exp(-5)\int_0^{\infty}\exp(-u)du = (1/2)\cdot\exp(-5).$

Similarly, $P_{MD}$ = P{missed detection} = (1/2)•exp(–5) also.

**(b)** The minimum-error-probability rule compares $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{\exp(-|u-5|)}{\exp(-|u+5|)} = \begin{cases} \exp(10), & u > 5, \\ \exp(2u), & -5 \le u \le 5, \\ \exp(-10), & u < -5, \end{cases}$

to $\frac{\pi_0}{\pi_1}$ which equals $\frac{1}{2}$ when $\pi_1 = 2\pi_0 = \frac{2}{3}$, and hence the minimum-error-probability rule chooses $H_1$ whenever $\exp(2\mathbf{X}) > 1/2$, i.e. $\mathbf{X} > -(1/2)\bullet\ln 2 = \theta$. Note that $-5 < \theta < 0$ and that $\exp(-\theta) = \sqrt{2}$.

$$P_{FA} = \int_\theta^\infty \frac{1}{2}\bullet\exp(-|u+5|)du = \int_\theta^\infty \frac{1}{2}\bullet\exp(-u-5)du = \frac{1}{2}\exp(-5)\int_\theta^\infty \exp(-u)du = \frac{1}{2}\bullet\exp(-5-\theta) = \frac{\exp(-5)}{\sqrt{2}}.$$ Similarly,

$$P_{MD} = \int_{-\infty}^\theta \frac{1}{2}\bullet\exp(-|u-5|)du = \int_{-\infty}^\theta \frac{1}{2}\bullet\exp(u-5)du = \frac{1}{2}\exp(-5)\int_{-\infty}^\theta \exp(u)du = \frac{1}{2}\bullet\exp(-5+\theta) = \frac{\exp(-5)}{2\sqrt{2}} = \frac{1}{2}P_{FA}.$$

Finally, the average error probability is $\pi_0 P_{FA} + \pi_1 P_{MD} = \sqrt{2}\bullet\exp(-5)/3$. More generally, the threshold $\theta$ equals $(1/2)\bullet\ln(\sqrt{\pi_0/\pi_1}$ and the average error probability is $\sqrt{\pi_0\pi_1}\exp(-5)$ which has maximum value (1/2))exp(–5) if $\pi_0 = \pi_1 = 1/2$. Of course, all the above applies only if $\exp(-10) < (\pi_0/\pi_1) < \exp(10)$.

**7.** The joint pdf has value 2 on the triangular region shown in the left-hand figure below.
For any a, $0 \le a < \infty$, $P\{\mathbf{Z} \le a\} = P\{\mathbf{Y}/\mathbf{X} \le a\} = P\{\mathbf{Y} \le a\mathbf{X}\} = P\{(\mathbf{X},\mathbf{Y})$ lies in the shaded region shown$\}$ $= 2\times((1/2)\times 1\times a/(a+1)) = a/(a+1)$. Hence, $f_{\mathbf{Z}}(a) = 1/(1+a)^2$ for $a \ge 0$ and 0 for $a < 0$.

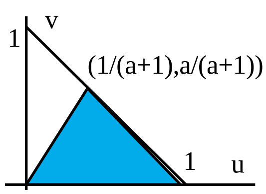
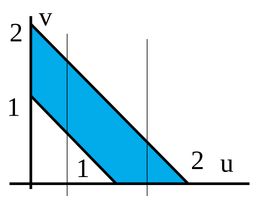
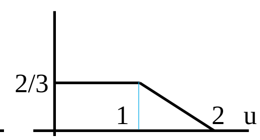


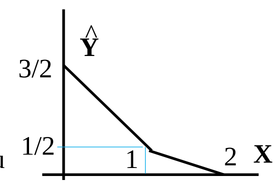
Figure for Problem 7



Problem 8: joint pdf



Problem 8: pdf of X



MMSE estimator

**8.** The joint pdf has value 2/3 on the region shown in the second figure above.

**(a)** $f_{\mathbf{X}}(u)$, the value of the marginal pdf of $\mathbf{X}$ at u, equals the cross-sectional area of the pdf surface at u. Hence, we get that $f_{\mathbf{X}}(u)$ has constant value 2/3 for $0 < u < 1$, and decreases to 0 as u increases from 1 to 2, as showb in the third figure above. More formally,

$$f_{\mathbf{X}}(u) = \begin{cases} 2/3 & \text{for } 0 < u < 1, \\ (2/3)\bullet(2-u) & \text{for } 1 \le u < 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that the area under the pdf is 1.

**(b)** The MMSE estimator for $\mathbf{Y}$ given $\mathbf{X}$ is the mean of the conditional pdf of $\mathbf{Y}$ given the value of $\mathbf{X}$. Also, the conditional pdf is simply the cross-section of the joint pdf surface "normalized" to have area 1.
It is easily seen that if $\mathbf{X} = a$ where $0 < a < 1$, then the conditional pdf of $\mathbf{Y}$ is uniform on $(1-a, 2-a)$ and hence has mean $(3/2) - a$ which varies from 3/2 at a = 0 to 1/2 at a = 1, while if $1 \le a < 2$, then the conditional pdf of $\mathbf{Y}$ is uniform on $(0, 2-a)$ and thus has mean $1 - a/2$ which varies from 1/2 at a = 1 to 0 at a = 2. Thus, the MMSE estimator $\hat{\mathbf{Y}}$ is $(3/2)-\mathbf{X}$ if $0 < \mathbf{X} < 1$, and $1 - \mathbf{X}/2$ if $1 \le \mathbf{X} < 2$, as illustrated in the right-hand figure above. Note that the function is *piecewise* linear, and thus is different from the linear MMSE estimator.

**9.(a)** $\mathbf{Z} = 5\mathbf{X} + \mathbf{Y}$ is a Gaussian random variable with mean $E[\mathbf{Z}] = E[5\mathbf{X} + \mathbf{Y}] = 5\times E[\mathbf{X}] + E[\mathbf{Y}] = 5\times 0 + 7 = 7$ and variance $\mathrm{var}(\mathbf{Z}) = 5^2\mathrm{var}(\mathbf{X}) + 1^2\mathrm{var}(\mathbf{Y}) + 2\times 5\times 1\times\mathrm{cov}(\mathbf{X},\mathbf{Y}) = 25\times 4 + 16 + 10\rho\sqrt{\mathrm{var}(\mathbf{X})\mathrm{var}(\mathbf{Y})}$ $= 100 + 16 + 10\times(1/16)\times 2\times 4 = 121 = 11^2$. Hence, $f_{\mathbf{Z}}(w) = 1/(11\sqrt{2\pi})\bullet\exp(-(w-7)^2/242)$, $-\infty < w < \infty$.

**(b)** $P\{\mathbf{Y} > 3\mathbf{X}\} = P\{3\mathbf{X} - \mathbf{Y} < 0\}$. But $3\mathbf{X} - \mathbf{Y}$ is a Gaussian random variable with mean $E[3\mathbf{X} - \mathbf{Y}] = 3\times E[\mathbf{X}] - E[\mathbf{Y}] = -7$ and variance $3^2\mathrm{var}(\mathbf{X}) + (-1)^2\mathrm{var}(\mathbf{Y}) + 2\times 3\times(-1)\times\mathrm{cov}(\mathbf{X},\mathbf{Y})$ $= 9\times 4 + 16 - 6\rho\sqrt{\mathrm{var}(\mathbf{X})\mathrm{var}(\mathbf{Y})} = 36 + 16 - 6\times(1/16)\times 2\times 4 = 49 = 7^2$. Hence, $3\mathbf{X} - \mathbf{Y}$ is $N(-7,7^2)$ and thus $P\{3\mathbf{X} - \mathbf{Y} < 0\} = \Phi\left(\frac{0-(-7)}{7}\right) = \Phi(1) = 0.8413$.