ECE 313 FINAL EXAMINATION

Friday May 3, 2002

Three hours

1. Let A, B, and C respectively denote the events that a student solves three problems (numbered 1, 2, and 3) on an exam. Assume that A, B, and C are independent events with probabilities 0.8, 0.9, and 0.7 respectively.

(a) What is the probability that the student solves at least one problem correctly?

(b) What is the probability that the student solves **only** Problem 2 correctly?

(c) Given that the student solved at least one problem correctly, what is the probability that Problem 3 was solved correctly?

2. A mailman starting a new route estimates that the probability that he is bitten by a dog on the very first day is 1/2. As each day passes without the mailman being bitten, he grows increasingly wary (and he becomes more cognisant of the locations of the dogs on his route). The *conditional* probability that the mailman is bitten on the n-th day, given that he has not been bitten on days 1 through n–1, is 1/(n+1). Let $\mathbf{X}$ denote the day on which the mailman is first bitten by a dog (at which time he goes postal, shoots the dog, and is transferred to a new route.)

(a) What is the pmf of $\mathbf{X}$? Hint: first find $P\{\mathbf{X} > n\}$ for n = 1, 2, …

(b) What is the expected value of $\mathbf{X}$?

3. Let $\mathbf{X}$ denote a geometric random variable. If $P\{\mathbf{X} = \text{even number}\} = \frac{4}{9}$, what is $E[\mathbf{X}]$?

4. Which of the following statements are true for **all** random variables $\mathbf{X}$ and $\mathbf{Y}$ with identical finite variance σ^2 ?

TRUE	FALSE	
☐	☐	$E[\mathbf{X}^2] = E[\mathbf{Y}^2]$
☐	☐	$\mathrm{var}(\mathbf{X} + \mathbf{Y}) = 2\sigma^2$
☐	☐	$\mathrm{var}(\mathbf{X} - \mathbf{Y}) = 0$
☐	☐	$\mathrm{var}(\mathbf{X} + \mathbf{Y}) + \mathrm{var}(\mathbf{X} - \mathbf{Y}) = 4\sigma^2$
☐	☐	$\mathrm{var}(2\mathbf{X} + 3\mathbf{Y}) = \mathrm{var}(3\mathbf{X} + 2\mathbf{Y})$
☐	☐	$\lvert\mathrm{cov}(\mathbf{X}, \mathbf{Y})\rvert \le \sigma^2$
☐	☐	$\mathbf{X} + \mathbf{Y}$ and $\mathbf{X} - \mathbf{Y}$ are uncorrelated random variables
☐	☐	$\mathbf{X} + \mathbf{Y}$ and $\mathbf{X} - \mathbf{Y}$ are independent random variables

5. Let $\mathbf{X}$ denote the time of the first arrival after t = 0 in a Poisson process with arrival rate λ, and let $\mathbf{Y} = \exp(\mathbf{X})$. Find $f_{\mathbf{Y}}(v)$, the pdf of $\mathbf{Y}$.

6. Under hypothesis H_0, a continuous random variable $\mathbf{X}$ has pdf $f_0(u) = (0.5)\exp(-|u + 5|)$, $-\infty < u < \infty$ while under hypothesis H_1, $\mathbf{X}$ has pdf $f_1(u) = (0.5)\exp(-|u - 5|)$, $-\infty < u < \infty$

(a) What is the maximum-likelihood decision rule, and what are the probabilities of *false alarm* (decision rule chooses H_1 when H_0 is true) and *missed detection* (decision rule chooses H_0 when H_1 is true)?

(b) Suppose that $P(H_0) = 1/3$ and $P(H_1) = 2/3$. Find the decision rule that minimizes the average error probability = P(decision rule chooses wrong hypothesis) and the corresponding minimum average error probability.

7. $\mathbf{X}$ and $\mathbf{Y}$ are random variables with joint pdf

$$f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} 2, & 0 < u < 1,\ 0 < v < 1,\ u + v < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $\mathbf{Z} = \mathbf{Y}/\mathbf{X}$.

8. The joint probability density function $f_{\mathbf{X},\mathbf{Y}}(u,v)$ for the continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ has constant value on the shaded region $\{(u, v) : 0 < u < 2, 0 < v < 2, 1 < u + v < 2$.

(a) Find $f_{\mathbf{X}}(u)$, the marginal probability density function for $\mathbf{X}$.

(b) What is $\hat{\mathbf{Y}}$, the minimum-mean-square-error (MMSE) estimator for $\mathbf{Y}$ given the value of $\mathbf{X}$? Your answer should be a function of $\mathbf{X}$.
If you prefer, you can draw a neat graph of this function but be sure that the graph is labeled completely.
Note that the problem is *not* asking for the *linear* MMSE estimator of $\mathbf{Y}$.

9. The jointly Gaussian random variables $\mathbf{X}$ and $\mathbf{Y}$ have means 0 and 7 respectively, variances 4 and 16 respectively, and correlation coefficient 1/16.

(a) Find the probability density function of $\mathbf{Z} = 5\mathbf{X} + \mathbf{Y}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{Z}}(w)$ for all real numbers w.

(b) Find the numerical value of $P\{\mathbf{Y} > 3\mathbf{X}\}$.