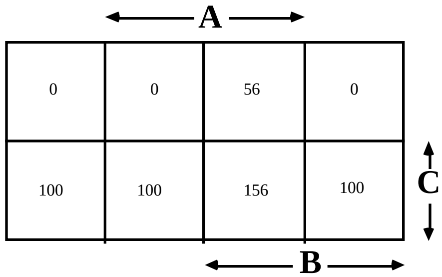
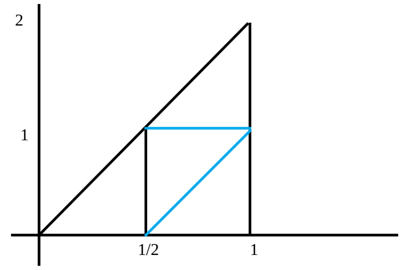
University of Illinois | **Solutions to Second Hour Exam** | **ECE 313 Spring 2002**

1. Let A, B, and C respectively denote the events that the U-O, O-C, and U-C links are in working condition. The capacities are marked on the Karnaugh map below (in which each cell has probability 1/8). It is easily seen that **Z** takes on values 0, 56, 100, and 156 with probabilities 3/8, 1/8, 3/8, and 1/8 respectively, and hence $E[\mathbf{Z}] = 0\times\frac{3}{8} + 56\times\frac{1}{8} + 100\times\frac{3}{8} + 156\times\frac{1}{8} = \frac{0 + 56 + 300 + 156}{8} = \frac{512}{8} = 64.$



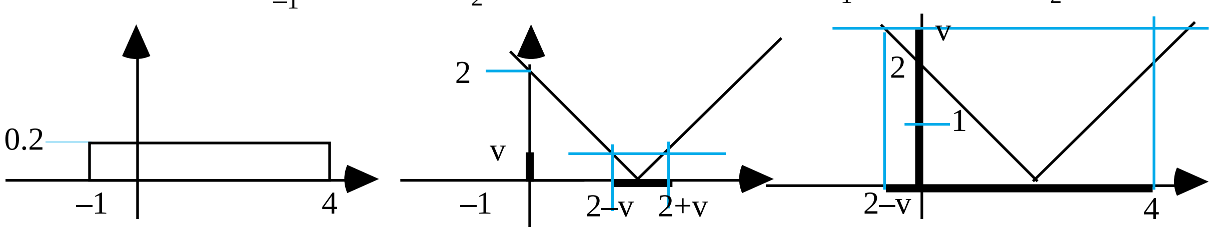


2. $1 = \int_{-\infty}^{\infty} f_{\mathbf{X}}(u)du = \int_0^1 a + bu\, du = au + bu^2/2\Big|_0^1 = a + b/2.$

Also, $\frac{2}{3} = E[\mathbf{X}] = \int_{-\infty}^{\infty} uf_{\mathbf{X}}(u)du = \int_0^1 au + bu^2\, du = au^2/2 + bu^3/3\Big|_0^1 = a/2 + b/3$. Hence, a = 0, b = 2. The pdf is thus 2u on [0, 1]. $P\{\mathbf{X} < 1/2\} = \int_0^{1/2} 2u\, du = \frac{1}{4}$. This is even easier if you sketch the pdf as shown.

3.(a) The pdf of **X** has value 0.2 for $-1 \le u \le 4$, is as shown below in the left-hand figure below. We have that

$E[\mathbf{Y}] = E[|\mathbf{X}-2|] = \int_{-1}^{2} 0.2(2-u)\, du + \int_2^4 0.2(u-2)\, du = 0.2(2u - u^2/2)\Big|_{-1}^{2} + 0.2(u^2/2-2u)\Big|_2^4 = 1.3.$



(b) **Y** takes on values in the range [0, 3]. From the middle and right-hand figures above, we see that for any v, $0 \le v \le 2$, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{2-v \le \mathbf{X} \le 2+v\} = 0.2(2+v-(2-v)) = 0.4v$, while for any v, $2 \le v \le 3$, $F_{\mathbf{Y}}(v) = P\{2-v \le \mathbf{Y} < 4\} = 0.2(4-(2-v)) = 0.2(2+v)$.

Hence, $f_{\mathbf{Y}}(v) = \begin{cases} 0.4, & 0 \le v \le 2, \\ 0.2, & 2 \le v \le 3 \\ 0, & \text{elsewhere} \end{cases}$ which is easily verified to be a valid pdf.

4.(a),(b) As studied in class, **X** is an exponential random variable with parameter μ. Hence, $P\{\mathbf{X} > \tau\} = \exp(-\mu\tau)$ and $f_{\mathbf{X}}(u) = \mu\bullet\exp(-\mu u)$ for u > 0, and 0 otherwise.

(c) $P\{\mathbf{X} > \tau \mid A\}$ is obviously 0 if $\tau \ge 3$, while for $0 \le \tau < 3$,

$P\{\mathbf{X} > \tau \mid A\} = P\{\text{No arrivals in } (0,\tau] \mid \text{two arrivals in } (0,3]\}$

$= P\{\text{No arrivals in } (0,\tau] \cap \text{two arrivals in } (0,3]\}/P\{\text{two arrivals in } (0,3]\}$

$= P\{\text{No arrivals in } (0,\tau] \cap \text{two arrivals in } (\tau,3]\}/P\{\text{two arrivals in } (0,3]\}$

$= [\exp(-\mu\tau)\bullet\exp(-\mu(3-\tau))\bullet(\mu(3-\tau))^2/2!]/\exp(-3\mu)\bullet(3\mu)^2/2! = (3-\tau)^2/9.$

(d) $f_{\mathbf{X}|A}(u|A) = -\frac{d}{d\tau} P\{\mathbf{X} > \tau \mid A\} = (6 - 2\tau)/9,\ \ 0 \le \tau < 3$, and 0 otherwise.

5. $P\{|\mathbf{X} - 4| > 3\} = P\{\mathbf{X} > 7\} + P\{\mathbf{X} < 1\} = 1 - \Phi\left(\frac{7-2}{5}\right) + \Phi\left(\frac{1-2}{5}\right) = 1 - \Phi(1) + \Phi(-0.2)$

$= 1 - \Phi(1) + 1 - \Phi(0.2) = 2 - 0.8413 - 0.5793 = 0.5794.$

$P\{\mathbf{X} < 3 \mid \mathbf{X} > 2\} = P\{2 < \mathbf{X} < 3\}/P\{\mathbf{X} > 2\} = 2\left[\Phi\left(\frac{3-2}{5}\right) - \Phi\left(\frac{2-2}{5}\right)\right] = 2(\Phi(0.2) - 0.5)$

$= 1.1586 - 1 = 0.1586$