1.(a) $P(A|B) = P(B|A)$ ⇔ $P(AB)/P(B) = P(AB)/P(A)$ ⇔ $P(A) = P(B)$ or $P(AB) = 0$ or both. Note that $P(AB) = 0$ does not necessarily mean that A and B are disjoint events. Thus, ´ is a true statement, but ` is not necessarily true. But ´ implies that $P(A^c) = P(B^c)$, so ˆ is a true statement too. Thus, none of the five choices given correctly describes which are the true statements.

(b) The value of a pdf can exceed 1 (e.g. uniform distribution on (a,b) with b–a < 1. Also, if ´ were true, the area under the pdf would be ∞ rather than 1. Thus, *Only* ` and ˆ are true statements.

(c) None of the five choices correctly describes which are the true statements.

2.

NO. $F_X(u) = 1 - F_X(-u)$ for all u, $-\infty < u < \infty$.

YES. $P\{\mathbf{X} > \alpha\} = 1 - F_X(\alpha) = F_X(-\alpha)$ for all α, $-\infty < \alpha < \infty$.

NO. $F_Y(v) = P\{\mathbf{Y} \le v\} = F_X(v) - F_X(-v)$ for $v \ge 0$, and $F_Y(v) = 0$ for $v < 0$.

YES. $f_Y(v)$, the derivative of $F_Y(v)$, is $f_X(v) + f_X(-v) = 2f_X(v)$ for $v \ge 0$, and 0 for $v < 0$.

NO. $F_X(-w)$ decreases from 1 to 0 as w increases from $-\infty$ to $+\infty$, and thus cannot be a CDF.

YES. $F_Z(w) = 1 - F_X(-w)$ has derivative $f_X(-w)$.

NO. A pdf cannot be negative…

YES. $E[\mathbf{X}] = 0$ by the symmetry of the pdf (Note: the mean exists because the variance is finite).

NO. $\mathrm{var}(\mathbf{Y}) = E[\mathbf{Y}^2] - (E[\mathbf{Y}])^2$. But, LOTUS gives $E[\mathbf{Y}^2] = E[\mathbf{X}^2] = \mathrm{var}(\mathbf{X}) = 9$ since $E[\mathbf{X}] = 0$. If $E[\mathbf{Y}] = 3$, then $\mathrm{var}(\mathbf{Y})$ would be 0… which it is not.

YES. LOTUS gives $E[\mathbf{Y}^2] = E[\mathbf{X}^2] = \mathrm{var}(\mathbf{X}) = 9$ since $E[\mathbf{X}] = 0$.

MAYBE. $\mathrm{var}(\mathbf{Y}) = E[\mathbf{Y}^2] - (E[\mathbf{Y}])^2$. Since $E[\mathbf{Y}] > 0$, $\mathrm{var}(\mathbf{Y})$ *can* be less than 8. For example, if $\mathbf{X}$ is uniformly distributed on $[-3\sqrt{3},+3\sqrt{3}]$ with variance $(6\sqrt{3})^2/12 = 9$, then $\mathbf{Y}$ is uniformly distributed on $[0,3\sqrt{3}]$ and thus $\mathrm{var}(\mathbf{Y}) = (3\sqrt{3})^2/12 = 2.25$.

YES. $\mathrm{var}(\mathbf{Y}) = E[\mathbf{Y}^2] - (E[\mathbf{Y}])^2 = 9 - (E[\mathbf{Y}])^2 < 9$ since $E[\mathbf{Y}] > 0$.

YES. Using LOTUS, $E[\mathbf{XY}] = \int_0^\infty u^2 f_X(u)\,du + \int_{-\infty}^0 -u^2 f_X(u)\,du = 0$.

YES. $\mathrm{cov}(\mathbf{X},\mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = 0$.

NO. Knowing the value of $\mathbf{X}$ tells you the value of $\mathbf{Y}$ exactly!

YES. By LOTUS, $E[\mathbf{XZ}] = E[-\mathbf{X}^2] = -E[\mathbf{X}^2] = -\mathrm{var}(\mathbf{X}) = -9$.

YES. $\mathbf{X} + \mathbf{Y} = \mathbf{X} + |\mathbf{X}| = \begin{cases} 0, & \text{if } \mathbf{X} \le 0, \\ 2\mathbf{X} > 0, & \text{if } \mathbf{X} > 0. \end{cases}$ Thus, $P\{\mathbf{X} + \mathbf{Y} \le 0\} = P\{\mathbf{X} \le 0\} = 1/2$.

YES. By Chebyshev's inequality, $F_Y(6) = F_X(6) - F_X(-6) = P\{-6 \le \mathbf{X} \le 6\} \ge 1 - (\sigma/6)^2 = 3/4$.

MAYBE. If $\mathbf{X}$ is a Gaussian random variable, then $P\{\mathbf{X} \le 6\} = \Phi(2) = 0.9772$ according to the table of values of $\Phi(x)$.

YES. $P\{\mathbf{X}^2 + 4\mathbf{X} + 3 < 0\} = P\{(\mathbf{X}+1)(\mathbf{X}+3) < 0\} = P\{-3 < \mathbf{X} < -1\} = F_X(-1) - F_X(-3)$.

YES. $P\{\mathbf{X}^2 - 4\mathbf{X} + 3 > 0\} = P\{(\mathbf{X}-1)(\mathbf{X}-3) > 0\} = P\{\mathbf{X} > 3\} + P\{\mathbf{X} < 1\} = 1 - F_X(3) + F_X(1)$ $= F_X(-3) + 1 - F_X(-1) = P\{\mathbf{X} \le -3\} + P\{\mathbf{X} > -1\} = P\{\mathbf{X}^2 + 4\mathbf{X} + 3 > 0\}$.

3. The Karnaugh map on the left is marked with letters a-h denoting the probabilities of the eight sets.

A

a	b	c	d	
e	f	g	h	C

B

A

0.2	0	0.2	0.1	
0.1	0.2	0.1	0.1	C

B

A

0.2	0.1	0.2	0	
0.1	0.1	0.1	0.2	C

B

Since (b+d+e) + (c+f+h) + g = 0.8, and (c+f+h)+g = 0.6,we readily obtain that a = 0.2 and b+d+e = 0.2. Next, adding together the three equations c+d+g+h = 0.5, b+c+f+g = 0.5, e+f+g+h = 0.5, we get (b+d+e) + 2(c+f+h) + 3g = 1.5, i.e. 2(c+f+h) + 3g = 1.3 which combined with (c+f+h)+g = 0.6 gives (c+f+h) = 0.5 and g = 0.1. Hence, P{all three events occurred} = g = 0.1, and P{all three | at least two} = g/(c+f+h+g) = 0.1/0.6 = 1/6. On the other hand, P{at least two | A} = (c+g+h)/(1/2) = 2(c+g+h) cannot be determined from the given information. For example, the two Karnaugh maps shown on the right are consistent with the given data, and give different values for the desired probability.

4. $P\left\{\sqrt{|1+\mathbf{X}|} > 4\right\} = P\{|1+\mathbf{X}| > 16\} = P\{\mathbf{X} > 15\} + P\{\mathbf{X} < -17\} = 1 - \Phi\left(\frac{15-3}{10}\right) + \Phi\left(\frac{-17-3}{10}\right)$
$= 1 - \Phi(1.2) + \Phi(-2) = 1 - \Phi(1.2) + 1 - \Phi(2) = 2 - 0.8849 - 0.9772 = 0.1379.$

5. Let $P\{\mathbf{X} = 1, \mathbf{Y} = 1\} = x$. Then, independence requires that $x = P\{\mathbf{X} = 1, \mathbf{Y} = 1\} = P\{\mathbf{X} = 1\}P\{\mathbf{Y} = 1\} = (x + 0.3)(x + 0.2)$, that is, $x = x^2 + 0.5x + 0.06$, i.e. $x^2 - 0.5x + 0.06 = (x - 0.2)(x - 0.3) = 0$. Hence, $P\{\mathbf{X} = 1, \mathbf{Y} = 1\}$ must have value 0.2 or 0.3, and it is easily shown that both values give independence, that is, $P\{\mathbf{X} = i, \mathbf{Y} = j\} = P\{\mathbf{X} = i\}P\{\mathbf{Y} = j\}$ for $i, j \in \{0,1\}$.

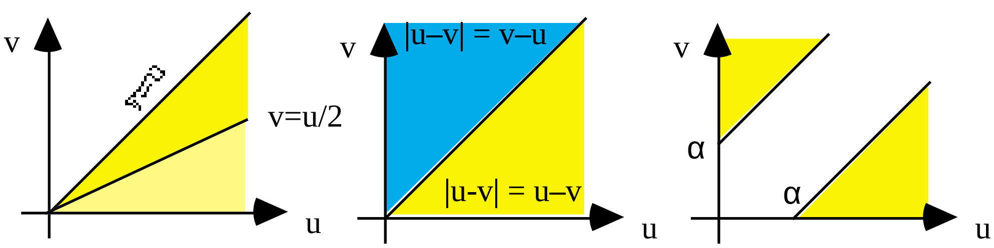
6. The joint pdf is nonzero on the triangular region shown below in the left-hand figure.

(a) Since the random point $(\mathbf{X},\mathbf{Y})$ always lies in the region $\{(u,v) : v < u\}$, it follows immediately that $P\{\mathbf{Y} < \mathbf{X}\} = 1$. We can also calculate

$$P\{2\mathbf{Y} < \mathbf{X}\} = P\{\mathbf{Y} < \mathbf{X}/2\} = \int_{u=0}^{\infty}\int_{v=0}^{u/2} \exp(-u)\,du = \int_{u=0}^{\infty} (u/2)\bullet\exp(-u)\,du = (1/2)\bullet\Gamma(2) = (1/2)\bullet 1\bullet\Gamma(1) = 1/2.$$

(b) More generally, for $0 < \alpha < 1$, $P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{Y}/\mathbf{X} \le \alpha\} = P\{\mathbf{Y} \le \alpha\mathbf{X}\} = \int_{u=0}^{\infty}\int_{v=0}^{\alpha u} \exp(-u)\,du = \alpha.$

(c) Since $F_{\mathbf{Z}}(\alpha) = \begin{cases} 0, & \alpha \le 0, \\ \alpha, & 0 < \alpha < 1, \\ 1, & \alpha \ge 1, \end{cases}$ we get that $f_{\mathbf{Z}}(\alpha) = \begin{cases} 1, & 0 < \alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$



7. $|u - v| = \begin{cases} u - v, & \text{if } u \ge v, \\ v - u, & \text{if } v > u. \end{cases}$ Hence, breaking the $E[\mathbf{Z}]$ integral into two and integrating by parts gives

$$E[\mathbf{Z}] = \int_{u=0}^{\infty}\int_{v=0}^{\infty} |u - v| \exp(-u-v)\,dv\,du = \int_{u=0}^{\infty}\int_{v=0}^{u} (u - v)\exp(-u-v)\,dv\,du + \int_{u=0}^{\infty}\int_{v=u}^{\infty} (v - u)\exp(-u-v)\,dv\,du$$

$$= \int_{u=0}^{\infty} \left[-u\bullet\exp(-v) + (1+v)\exp(-v)\right]_0^u + \left[-(1+v)\bullet\exp(-v) + u\bullet\exp(-v)\right]_u^{\infty} du$$

$$= \int_{u=0}^{\infty} 2\bullet\exp(-2u) + u\bullet\exp(-u) - \exp(-u)\,du = 1 + 1 - 1 = 1$$

on recognizing the integrands as the pdfs of gamma random variables with parameters (1,2), (2,1) and (1,1) respectively (or working out the integrals; they are not hard!).

$E[\mathbf{Z}^2] = E[|\mathbf{X} - \mathbf{Y}|^2] = E[(\mathbf{X} - \mathbf{Y})^2] = E[\mathbf{X}^2] + E[\mathbf{Y}^2] - 2E[\mathbf{XY}]$. But, $E[\mathbf{X}^2] = \mathrm{var}(\mathbf{X}) + (E[\mathbf{X}])^2 = 1 + 1$, $E[\mathbf{Y}^2] = \mathrm{var}(\mathbf{Y}) + (E[\mathbf{Y}])^2 = 1 + 1$, and $E[\mathbf{XY}] = E[\mathbf{X}]E[\mathbf{Y}] = 1$ since $\mathbf{X}$ and $\mathbf{Y}$ are independent (and hence uncorrelated). Thus, $E[\mathbf{Z}^2] = 2 + 2 - 2 = 2$, and $\mathrm{var}(\mathbf{Z}) = E[\mathbf{Z}^2] - (E[\mathbf{Z}])^2 = 2 - 1 = 1$.

It is also possible to compute the pdf of $\mathbf{Z}$ and deduce the mean and variance from this. For $\alpha > 0$, $P\{\mathbf{Z} > \alpha\} = 1 - F_{\mathbf{Z}}(\alpha) = P\{\mathbf{X} - \mathbf{Y} > \alpha\} + P\{\mathbf{X} - \mathbf{Y} < -\alpha\} = P\{(\mathbf{X},\mathbf{Y}) \in \text{shaded region in rightmost figure}\}$

$$= 2\int_{u=\alpha}^{\infty}\int_{v=0}^{u-\alpha} \exp(-u-v)\,dv\,du = 2\int_{u=\alpha}^{\infty} \exp(-u)[1 - \exp(-u+\alpha)]\,du = 2[\exp(-\alpha) - (1/2)\exp(-\alpha)] = \exp(-\alpha).$$

Hence, $\mathbf{Z}$ is an exponential random variable with parameter 1 !! and therefore $E[\mathbf{Z}] = 1$ and $\mathrm{var}(\mathbf{Z}) = 1$ as obtained above.