University of Illinois — **Problem Set #14** — **ECE 313**

Spring 1999

Assigned: Wednesday, April 28, 1999
Due: Wednesday, May 5, 1999
Reading: Chapter 7 (Sections 1-3 only), Chapter 8
Problems:

1. If **X** is $N(0,\sigma^2)$, then $\mathbf{X}^2$ has gamma pdf with parameter $(1/2, 1/2\sigma^2)$. We did the case for $\sigma^2 = 1$ in class, and the general case is very similar. Now, suppose that **X**, **Y**, and **Z** are independent $N(0,\sigma^2)$ random variables. Then $\mathbf{X}^2$, $\mathbf{Y}^2$, and $\mathbf{Z}^2$ are independent gamma random variables with parameter $(1/2, 1/2\sigma^2)$.

(a) Use the comment immediately following the proof of Proposition 3.1 (pp. 266-267) of Ross to *state* what the *type* of pdf of $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$, and write down *explicitly* the exact pdf. What *is* the numerical value of $f_{\mathbf{W}}(5)$ if $\sigma^2 = 4$?

(b) Use LOTUS to prove that $E[\mathbf{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer, shame on you!

(c) In a physical application, **X**, **Y**, and **Z** represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m as . Thus, $\mathbf{H} = (1/2)m\mathbf{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in °K. (Note that the average energy is (1/2)kT per dimension.) Show that the kinetic energy **H** has the Maxwell-Boltzmann pdf $f_{\mathbf{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\exp(-\beta/kT)$, $\beta > 0$.

(d) $\mathbf{V} = \sqrt{\mathbf{W}} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$ is the "speed" of the molecule. Show that the pdf of **V** is $f_{\mathbf{V}}(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\gamma^2\exp\left(\frac{m\gamma^2}{2kT}\right)$ $\gamma > 0$ cf. Theoretical Exercise 1, p. 237 of the text.

(e) What is the average speed of the molecule?

2. Let $(\mathbf{X},\mathbf{Y})$ be uniformly distributed on the unit disc (radius = 1) centered at the origin. Find the expected value of the distance from the origin to the point $(\mathbf{X},\mathbf{Y})$.

3. Except for the trivial case when all the probability mass is at $\mu = E[\mathbf{X}]$, there is probability mass both to the left and right of μ; in particular, there is an ω Ω such that $\mathbf{X}(\omega) < \mu$. Is it also true that if $(E[\mathbf{X}], E[\mathbf{Y}]) = (\mu_1, \mu_2)$, then there is an ω Ω such that $\mathbf{X}(\omega) < \mu_1$ and $\mathbf{Y}(\omega) < \mu_2$? If you believe the result is true, prove it. Otherwise, give a counterexample to show that it is false.

4. Let the random variables **X** and **Y** be independent and uniformly distributed on (0,1). Find $E(|\mathbf{X}-\mathbf{Y}|)$ and $\mathrm{Var}(\mathbf{X}-\mathbf{Y})$.

5. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\mathrm{var}(\mathbf{X}) = 4$, $\mathrm{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$

(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\mathrm{cov}(\mathbf{T}, \mathbf{U})$?

(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\mathrm{var}(\mathbf{W})$.

(d) If **X** and **Y** are jointly Gaussian random variables, and **W** is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

6.(a) If $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\mathrm{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\mathrm{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\mathrm{var}(\mathbf{X}) = 3\bullet\mathrm{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?

(b) If instead of having values 36 and 64, $\mathrm{var}(\mathbf{X}) = \mathrm{var}(\mathbf{Y})$, are **X** and **Y** uncorrelated ?