Assigned: Wednesday, April 21, 1999
Due: Wednesday, April 28, 1999
Reading: Ross, Chapter 6 (except Sections 6.6 and 6.8), Chapter 7 (Sections 1-3 only)
Noncredit Exercises: pp. 293-300: 8-12, 15, 20-23, 26, 28-30, 41-43, 51, 54;
pp. 300-304: 8, 14, 22, 23, 33; pp. 372-384: 1, 16, 26, 29, 34, 36;
pp. 384-392: 1, 2, 17, 22, 23, 40; pp. 421-424: 1-9, 15
Problems:

1. Let ($\mathbf{X}$, $\mathbf{Y}$) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) What is the value of C?

(b) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 0.25\}$.

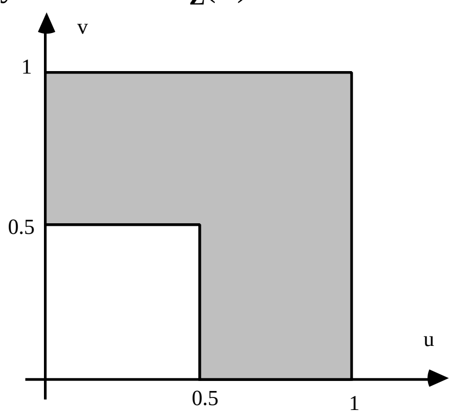
2. The random point ($\mathbf{X}$,$\mathbf{Y}$) is uniformly distributed on the shaded region shown in the left-hand figure below.

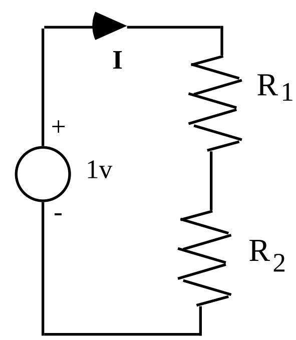
(a) What is the numerical value of $f_{\mathbf{X},\mathbf{Y}}(0.75,0.75)$?

(b) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{X}}(u)$ for all real numbers u, $-\infty < u < \infty$.

(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.

(d) Find the pdf of the random variable $\mathbf{Z} = \mathbf{Y}/\mathbf{X}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{Z}}(\alpha)$ for all real numbers α, $-\infty < \alpha < \infty$.





3. Two resistors are connected in series to a one-volt voltage source as shown in the right-hand diagram above. Suppose that the resistance values $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval (0, 1). Find the pdf $f_{\mathbf{I}}(a)$ of the current $\mathbf{I}$ (measured in amperes) in the circuit. Be sure to specify the value of $f_{\mathbf{I}}(a)$ for all real numbers a, $-\infty < a < \infty$.

4. Let ($\mathbf{X}$, $\mathbf{Y}$) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$
Find the pdf of $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$.

5. Two random break-points $\mathbf{X}_1$ and $\mathbf{X}_2$ are chosen on a stick of unit length, thereby breaking the stick into three pieces. Thus, if $\mathbf{X}_2 > \mathbf{X}_1$, the three pieces have lengths $\mathbf{X}_1$, $\mathbf{X}_2 - \mathbf{X}_1$, and $1 - \mathbf{X}_2$. If $\mathbf{X}_1 > \mathbf{X}_2$, interchange $\mathbf{X}_1$ and $\mathbf{X}_2$ in the above. Luckily, $P\{\mathbf{X}_2 = \mathbf{X}_1\} = 0$. In this problem, you will compute the probability that the three pieces can form a triangle based on various assumptions.

(a) Sketch the u-v plane and indicate on it the region(s) such that if the random point ($\mathbf{X}_1$, $\mathbf{X}_2$) lies in the region(s), then the pieces can form a triangle. These regions will come in handy in what follows.

(b) Assume that the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and uniformly distributed on (0,1). Show that the probability that the three pieces can form a triangle is 1/4.

(c) What is the probability that the three pieces can form a triangle if the joint pdf of $\mathbf{X}_1$ and $\mathbf{X}_2$ is given by $f_{\mathbf{X}_1,\mathbf{X}_2}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

(d) Suppose that $\mathbf{X}_1$ is uniformly distributed on (0,1) and that the *conditional* density of $\mathbf{X}_2$ given $\mathbf{X}_1$ = u is uniform on (u,1), that is, we break the stick at $\mathbf{X}_1$ (choosing the break-point with uniform density), and then break the right-hand piece at $\mathbf{X}_2$ (choosing the second break-point with uniform density on the *right-hand piece).*
Let T denote the event that the pieces can form a triangle. Find the *conditional probability* $P(T|\mathbf{X}_1 = u)$, for 0 < u < 1. (Hint: you will find that different expressions apply depending on the value of u). Now find the *unconditional probability* using the result that

$$P(T) = \int P(T|\mathbf{X}_1 = u) f_{\mathbf{X}_1}(u)du$$

where the integral is over the range (-∞, ∞) in general (but only over (0,1) in this case).

(e) With the same assumptions as in (c), now find the joint pdf of $\mathbf{X}_1$ and $\mathbf{X}_2$. Show that this joint pdf is nonzero only over the triangular region 0 < u < v < 1. Use this pdf and the regions in part (a) to find the probability that the pieces can form a triangle. You should get the same answer as in part (c).

(f) From the joint pdf in (d), compute the marginal pdfs of $\mathbf{X}_1$ and $\mathbf{X}_2$. The marginal pdf of $\mathbf{X}_1$ should be uniform on (0,1). (It isn't? Better check your work!). **Prove** that the pdf that you obtain for $\mathbf{X}_2$ is a valid pdf.

Exercise: ("Sticks and stones may break my bones, but more exercises in probability can never hurt me!") What if after breaking the stick at $\mathbf{X}_1$, we pick one of the pieces at random and break *it* at random? Thus, given $\mathbf{X}_1$ = u, the conditional pdf of $\mathbf{X}_2$ is uniform on (0, u) or (u, 1) depending on which piece is picked. What is P(T) in this case?