Assigned: Friday, April 9, 1999
Due: Wednesday, April 14, 1999

Reminder: **Hour Exam II** is scheduled for **Wednesday April 14, 7:00 pm to 8:00 pm,** in **Room 269 Everitt Laboratory.**

One 8.5" by 11" sheet of notes is allowed.

Calculators, laptop computers, Palm Pilots etc are not allowed.

The material covered on Problem Sets 5-10 and through the lecture of April 5 (the Poisson process) is included on the exam. *This* Problem Set also has material that will give you practice in working with random variables and thus will help you prepare for the exam.

Coverage of material from Ross is as follows: Chapter 3-5, except for Sections 4.7.2, 4.8.1, 4.9.4, 5.4.1, 5.6.2, and 5.6.4; Chapter 8 upto the bottom of page 397 (Chebyschev inequality); Chapter 9.1 (Poisson process) ; and Chapter 10.2.1 (simulation of random variables).

Reading: Ross, Chapter 6 (except Sections 6.6 and 6.8), Chapter 7 (Sections 1-3 only)
Noncredit Exercises: pp. 293-300: 8-12, 15; pp. 300-304: 8, 14;
Problems:

1. A VLSI chip has constant hazard rate $\lambda = -\ln 0.999$/week.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the module lasts for at least one week?

Now suppose that three identical chips are organized into a triple–modular–redundancy (TMR) system in which we assume that the majority–logic gate cannot fail. Furthermore, we assume that the three chips fail independently of one another, that is, if their lifetimes are $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, then the events $\{\mathbf{X}_1 > t_1\}$, $\{\mathbf{X}_2 > t_2\}$, and $\{\mathbf{X}_3 > t_3\}$ are independent for all t_1, t_2, t_3. Let $\mathbf{Y}$ denote the length of time for which the TMR system functions correctly.

(c) If you are told that the event $\{\mathbf{Y} > t\}$ occurred, what can you say about the occurence (or nonoccurence) of the events $\{\mathbf{X}_1 > t\}$, $\{\mathbf{X}_2 > t\}$, and $\{\mathbf{X}_3 > t\}$?

(d) Show that $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

(e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?

(f) Find t such that $P\{\mathbf{Y} > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?

2. Let $\mathbf{X}$ denote a unit Gaussian random variable.

(a) Find $E[|\mathbf{X}|]$.

(b) What is the conditional pdf of $\mathbf{X}$ given that $\mathbf{X} > 0$? i.e., what is $f_{\mathbf{X}|\mathbf{X}>0}(u|\mathbf{X} > 0)$?

(c) Suppose that $\mathbf{Y}$ is a random variable whose pdf just happens to be exactly the pdf you found in part (b). What is $E[\mathbf{Y}]$?

3. If hypothesis H_0 is true, the pdf of **X** is $f_0(u) = (1/2)\exp(-|u+1|)$, $-\infty < u < \infty$, while if hypothesis H_1 is true, the pdf of **X** is $f_1(u) = (1/2)\exp(-|u-1|)$, $-\infty < u < \infty$.
Such pdfs (which we also encountered in Problem 3 of Problem Set #10) are called LaPlacian or double exponential pdfs (cf. Ross, p.219 and Example 5e)

(a) Sketch the two pdfs.
Be careful: those absolute-value signs are trickier than they look!

(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable **X** instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ with 1.

(c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part (b)?

(d) Compute the values of the likelihood ratio for $u = -4, -3, -2, \ldots, 2, 3, 4$.

(e) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to (π_0/π_1). Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable **X**.

(f) If $\pi_0 = 2\pi_1$, what is the average probability of error of the Bayesian decision rule?

(g) What is the average error probability of a decision rule that always decides H_0 is the true hypothesis, regardless of the value taken on by **X**?

(h) Show thar if $\pi_0 > e^2/(e^2+1)$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by **X**.

4. The random variable **X** models a physical parameter. If hypothesis H_0 is true, then, $f_0(u)$, the pdf of **X**, is Gaussian with mean 0 and variance a^2. On the other hand, if hypothesis H_1 is true, then $f_1(u)$, the pdf of **X**, is Gaussian with mean 0 and variance $b^2 > a^2$.

(a) Suppose that H_0 and H_1 have equal probability. Thus, for i = 0, 1, the pdf of **X** when hypothesis H_i is true can be thought of as the *conditional* pdf of **X** given that H_i occurred, i.e. $f_{\mathbf{X}|H_i}(u|H_i)$. Write an expression for the *unconditional* pdf of **X**. Is the unconditional pdf of **X** a Gaussian pdf?

(b) What is the likelihood ratio? Simplify your answer.

(c) What is the maximum-likelihood decision rule, and what are the false alarm probability and the missed detection probability of this rule?

(d) Consider all possible decision rules of the form: "Choose H_0 if $|\mathbf{X}| < \theta$, and choose H_1 otherwise." Note that the various rules differ only in the threshold θ, so that specifying the threshold specifies a decision rule. For what values of θ is the decision rule such that P(false alarm) ≤ 0.05? Of all the decision rules that satisfy P(false alarm) ≤ 0.05, which one has the smallest P(missed detection)?