Assigned: Wednesday, March 31, 1999
Due: **Friday**, April 9, 1999
Reading: Ross, Chapter 5, and Appendix of my Lecture Notes for ECE 313
Reminders: There is no class on Wednesday April 7 (time given off in lieu of the upcoming evening Hour Exam: Wednesday April 14, 7:00 pm — 8:00 pm in 269 EL)
Noncredit Exercises: Ross, pp. 173-184: 17-19, 40-43, 49-56 70, 73; pp. 184-188: 5, 25-28; pp. 232-237: 30, 31, 35-39; pp. 237-241: 14, 28, 29.
Problems:

1. A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where **Y** = α if **X** > 0 and **Y** = $-\alpha$ if **X** $\le$ 0) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X}-\mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X}-\alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X}+\alpha)^2 & \text{if } \mathbf{X} \le 0. \end{cases}$

So we want to choose α so that E[**Z**] is as small as possible. Use LOTUS to easily find E[**Z**] as a function of α, and then find the value of α that minimizes E[**Z**].

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, **W** = 3 if **X** $\ge$ 2.5, **W** = 2 if 1.5 $\le$ **X** < 2.5, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is $(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_0)$. What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) Noncredit exercise (but a real-life engineering problem!): Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X}-\mathbf{W})^2]$?

2. [Read Example 3d on pp. 203-204 first.] Let the (straight) line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length **X** of the *arc* (measured clockwise around the circle) is a random variable uniformly distributed on $[0,2\pi)$. Now consider the "random chord" AD.

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express **L** as a function of the random variable **X**, and find the probability density function for **L**.

3. **X** is a continuous random variable with pdf $f_X(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.

(a) What is the value of P{**X** $\le$ ln 2}?

(b) Find the conditional probability that P{|**X**| $\le$ ln 2} given that {**X** $\le$ ln 2}.

(c) Now suppose that **X** denotes the voltage applied to a semiconductor diode, and that the current **Y** is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of **Y**.

4. Raw scores on the SAT (and GRE) are transformed by a nonlinear function so that the minimum score is 200 and the maximum is 800. The histogram of scores *resembles* a Gaussian pdf with mean 500 and variance ß$^2 = 100^2$, that is, the score **X** of a student chosen at random can be modeled as a Gaussian random variable with mean 500 and variance $\sigma^2 = 100^2$. According to this model,

(a) what should your percentile rank be if your score is 700?

(b) what score corresponds to a percentile rank of 95% ?

(c) What fraction of students score between 300 and 550?

5. Let $Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left(-\frac{u^2}{2}\right)du = 1 - \Phi(x)$ where $\Phi(x)$ denotes the CDF of a unit Gaussian random variable.

(a) Some tables list the values of $Q(x)$ (instead of $\Phi(x)$) for large values of x. Why might the tabulator have chosen to specify $Q(x)$ instead of $\Phi(x)$? Explain briefly.

On page 211 (p. 218 in 4th edition), Ross gives an upper and a lower bound on $Q(x)$ (Eq. (4.4)). The rest of this problem leads you through a derivation of Eq. (4.4) that does not use the "obvious inequality" invoked by Ross in his proof, and it also looks at another, simpler bound.

(b) What is the derivative of $\exp(-u^2/2)$ with respect to u?

(c) Write the integrand for $Q(x)$ as $(\sqrt{2\pi})^{-1}u^{-1}(u\exp(-u^2/2))$ and integrate by parts to deduce the upper bound on $Q(x)$. Repeat the trick of re–writing and integrating by parts to deduce the lower bound on $Q(x)$. Are these bounds useful as $x \to 0$? Why or why not? What is the asymptotic value of the ratio of the bounds as $x \to \infty$?

(d) A useful bound when x is small is $Q(x) \le (1/2)\exp(-x^2/2)$ for $x \ge 0$ in which equality holds only at x = 0. Derive this bound by first showing that $t^2 - x^2 > (t - x)^2$ for $t > x > 0$ and then applying this result to $\exp(x^2/2)Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left[-\left(\frac{t^2 - x^2}{2}\right)\right]dt$

(e) For what values of x is this smaller than the upper bound of Eq.(4.4)?

6. Do **either** part (a) **or** part (b). Then do parts (c)–(e).

(a) Attach to your homework a **photocopy** of your calculator's manual page(s) that explains which **formula** your calculator computes Q(x). Reading the page might help too!
Note: I **do not want** to know **which buttons** you have to press in order to find Q(x); I **want** to know **what formula** your calculator uses internally to find Q(x).
The xerographically–challenged are permitted to just copy the relevant formulas to their homework. **NEXT:** press the appropriate buttons to find Q(5).
If your calculator cannot compute Q(x), or if the manual does not state what formula is used to calculate Q(x) but just tells you which buttons to press, or if you have lost the manual, do part (b) instead.

(b) Read Chapter 26.2 of Abramowitz and Stegun (*reference* book *(not a **reserve** book)* in Grainger Engineering Library), and use Equation 26.2.17 to calculate Q(5).

(c) The number found in part (a) or (b) is just an *approximation* to the value of Q(5). Use the maximum error specification to find the *range* in which the actual value of Q(5) must necessarily lie. What is the *maximum relative error* in the approximation to Q(5) that you found in part (a) or (b)? Note: the relative error is defined as $\frac{|\text{true value} - \text{computed value}|}{\text{true value}}$ expressed as a percentage.

(d) On p. 972, Abramowitz and Stegun give the value of $-\log_{10}Q(5)$. Blindly trust your calculator to do the exponentiation correctly and find the *actual relative error* in the approximation to Q(5) that you found in part (a) or (b). What would the actual relative error have been if you had simply used the upper bound of Eq. (4.4) as an approximation to Q(5) as suggested by Ross? What if you had ignored Ross's suggestion and used the lower bound as an approximation to Q(5) instead?

(e) Explain why the "much easier" Equation 26.2.18 of Abramowitz and Stegun is not particularly useful for computing Q(5).

7. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable **X** with pdf $f(u) = (bu)\cdot\exp(-bu^2/2)$ for $u > 0$ (Ross, p. 221). The complementary CDF is given by $P\{\mathbf{X} > t\} = \exp(-bt^2/2)$ for $t > 0$.

(a) Find the mean lifetime E[**X**] of the system using the formula $E[\mathbf{X}] = \int_0^\infty u\cdot f(u)\,du$.

(b) Use the result $E[\mathbf{X}] = \int_0^\infty P\{\mathbf{X} > t\}\,dt$ to find the mean lifetime of the system. Do you get the same answer as in part (a)? Why or why not?

(c) What is the median lifetime, and is it larger or smaller than the mean lifetime? How do these parameters compare to the mode of lifetime (mode = location of the pdf maximum)

(d) The system fails at time t, i.e. **X** = t is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?