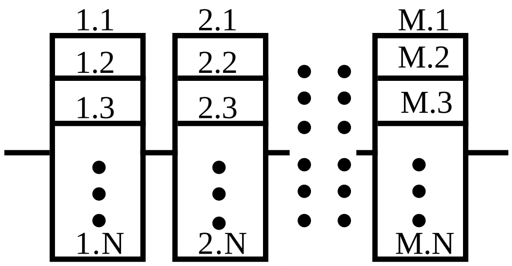
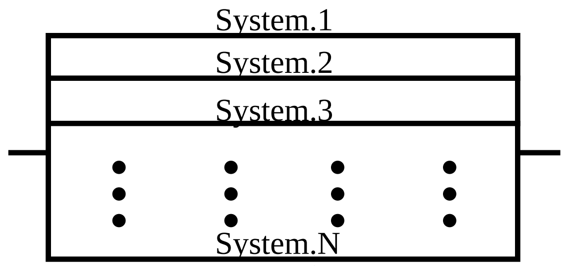
Assigned: Wednesday, March 3, 1999
Due: Wednesday, March 10, 1999
Reading: Ross, Chapters 4 and 5
Noncredit Exercises: Ross, pp. 173-184: 17, 19, 22, 33, 36, 37, 39, 44, 45;
pp. 184-188: 9, 20; pp. 189-191: 2, 9, 11, 16, 17; pp. 232-237: 1-8; pp. 237-241: 1, 8

Problems:

1. A system works if and only if all its M subsystems (numbered 1 through M) work. Each subsystem fails (independently) with probability p. A more reliable system can be obtained by replicating each subsystem N times as shown in the graph model on the left. Or, we can replicate the entire system N times as shown in the graph model on the right.



N-ary replication of each subsystem



N-ary replication of entire system

(a) For each model, find the probability of (replicated) system failure in terms of p, N and M.

(b) Suppose that M = 5 and p = 0.2. If it is desired that the (replicated) system failure probability be less than 0.001, what should N be in each case?

(c) Repeat part (b) assuming that there are M subsystems numbered I, II, III, IV, … , M. Hint: This homework is due on March X, MIM.

2. Barney and Betty play a game in which they take turns tossing a fair coin. The first one to toss a head wins the game. Barney tosses first in each game.

(a) What is the probability that Barney wins? that Betty wins?

(b) Betty soon tires of this game and suggests that it would be more fair for the loser of a game to toss first in starting the next game. Once again, Barney tosses first in the *first* game, so that his win probability is the same as in part (a). Now, let p_n and $q_n = 1 - p_n$ denote the probabilities that Barney and Betty respectively win the n-th game. We have just decided that p_1 and q_1 are as in part (a). What are p_2 and q_2? What are p_3 and q_3?

(c) More generally, show that $p_n = (2/3) - (1/3)p_{n-1}$ and $q_n = (2/3) - (1/3)q_{n-1}$ and use these difference equations to find p_n and q_n in terms of p_1 and q_1. (If you never learned in Math 285 (or ECE 310) how to solve difference equations, assume that **for all values of n,** p_n can be expressed as $a + b\alpha^n$, substitute, and solve for α and a; the value of b is obtained from the "initial condition" p_1. Repeat for q_n — it is the same difference equation but the "initial condition" q_1 is different.)

(d) What is the limit as $n \to \infty$ of p_n and q_n ? Is this game asymptotically fair?

(e) Barney now proposes that instead of the first one to toss a head winning the game, the first one who *matches* the previous toss (whether the previous toss is part of the current game or the last toss of the previous game) wins. Betty accepts but generously insists that, as before, Barney still toss first (so that the poor schmuck has no previous toss to match on his first toss!). What are p_1 and q_1 now? p_2 and q_2? p_3 and q_3? Is this game asymptotically fair? Assume as in part (b) that the loser of one game tosses first in the next game.

3. The dice game of craps (see Ross, p. 58) begins with the player (called the shooter) rolling two fair dice. If the result is a 2, or 3, or 12, the shooter loses, while if the result is a 7 or 11, the shooter wins.

(a) What is the probability that the shooter loses on the first roll? What is the probability that the shooter wins on the first roll?

(b) If the sum of the dice on the **first roll** is any of 4, 5, 6, 8, 9, 10, that number is called the **shooter's point.** For **each** number i in the set {4, 5, 6, 8, 9, 10}, find the probability that the shooter's point is i. I need six answers here, folks!

(c) Suppose that the shooter's point is i where i is some number in {4, 5, 6, 8, 9, 10}. The shooter now rolls the two dice again. If the result is a 7, the shooter loses (craps out.) If the result is i, the shooter wins (this is referred to as making the point). If the result is neither i nor 7, the shooter rolls again. This process continues until the shooter either makes the point or craps out. Given that the shooter's point is i, what is the conditional probability that the shooter makes the point? Naturally, the answer depends on i, so here too, I need six answers.

(d) Use the above results to compute the probability of winning at craps.

(e) Given that the shooter's point is 8, what is the probability that the shooter makes it "the hard way," that is, by rolling two fours? Generally, bets are offered at 10-to-1 odds that the shooter does not make the point 8 the hard way. That is, if you bet $1, you win $10 (plus your $1 back!) if the shooter makes 8 the hard way; and you lose the $1 that you bet if the shooter loses out or makes 8 by rolling 2-6, 3-5, 5-3, or 6-2). In the long run over many such bets, do you expect to make money, or lose money, or come out even?

4. Let the random variable $\mathbf{I}_D$ denote the indicator function of an event D, that is,

$$\mathbf{I}_D(\omega) = \begin{cases} 1 & \text{if } \omega \quad D, \\ 0 & \text{if } \omega \quad D. \end{cases}$$

Let A, B, and C denote independent events with probability 1/2, and define the random variable $\mathbf{X}$ by $\mathbf{X}(\omega) = \mathbf{I}_A(\omega) + 2\mathbf{I}_B(\omega) - \mathbf{I}_C(\omega)$.

(a) What are the values taken on by the random variable $\mathbf{X}$?

(b) Find the cumulative probability distribution function $F_{\mathbf{X}}(u)$ and the probability mass function $p_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$. Be very careful in specifying the values of $F_{\mathbf{X}}(u)$ at points where the function is discontinuous.

5. Which of the following are valid cumulative probability distribution functions (cdfs) ? For those that are not valid cdfs, state at least one property of cdfs which is not satisfied. For those which are valid cdfs, compute $P\{|\mathbf{X}| > 0.5\}$.

(a) $$F_{\mathbf{X}}(u) = \begin{cases} 0, & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$$

(b) $$F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), & u < 0, \\ 1 - (1/4)\exp(-3u), & u \ge 0. \end{cases}$$

(c) $$F_{\mathbf{X}}(u) = \begin{cases} (1/2)\exp(2u), & u \le 0, \\ 1 - (1/4)\exp(-3u), & u > 0. \end{cases}$$

6. Let $\mathbf{X}$ denote the number of hours that a student works on ECE 340 each week. It is known that $\mathbf{X}$ is a mixed random variable with cumulative probability distribution function $F_{\mathbf{X}}(u)$ given by

$$F_{\mathbf{X}}(u) = \begin{cases} 0, & u < 0, \\ (1+u)/8, & 0 \le u < 1, \\ 1/2, & 1 \le u < 2, \\ 1/2 + u/8, & 2 \le u < 4, \\ 1, & u \ge 4. \end{cases}$$

Find the probability that the student

(a) works for exactly 2 hours,

(b) works for more than 2 hours,

(c) works for less than 2 hours,

(d) works for exactly 3 hours,

(e) works for more than 1/2 but less than 3 hours,

(f) works for more than 2 hours given that the student works at all, i.e. find $P\{\mathbf{X} > 2 | \mathbf{X} > 0\}$.