Assigned: Wednesday, February 24, 1999
Due: Wednesday, March 3, 1999
Reading: Ross, Chapter 3
Noncredit Exercises: Ross pp. 104-117: 53, 58, 59, 62, 63, 70-74, 78, 81

Reminder: **Hour Exam I** is scheduled for **Wednesday March 3, 7:00 pm to 8:00 pm,** in **Room 269 Everitt Laboratory.**

One 8.5" by 11" sheet of notes is allowed.

Calculators, laptop computers, Palm Pilots etc are not allowed.

The material covered on Problem Sets 2-5 is included on the exam. *This* Problem Set also has material on decision theory, and thus, working these problems will help you prepare for the exam. Coverage of material from Ross is as follows:

Chapter 1 (except Section 1.6)

Chapter 2 (except Section 2.6)

Chapter 3 (except for Section 3.4 on independent events. However, note that we considered Example 4f on pp. 86-87 in the context of random variables)

Chapter 4.1, 4.3–4.5, 4.7 upto and including Proposition 7.1 on p. 150, 4.8 through p. 156, and 4.9.1-4.9.3. Material on CDF and variance is *not included*. However, you must know (or have on your sheet of notes) the pmfs and means of the binomial, Poisson, and geometric random variables.

Additional material (not always in Ross but covered in class and on homework) can be found in my Lecture Notes in Chapters 1, 2 (except for decision making involving costs on pp. 32-33), pp. 54-59 of Chapter 3. Material on random variables can be found on pp. 63-67, 71-72, and 92-95 of Chapter 4.

Problems:

1. ["I'm leaving on a prop plane"] Consider again Problem #2 of Problem Set #4. Suppose that 15 of the 105 passengers who hold reservations are arriving in Chicago on a connecting flight. If the connecting flight is on time, all 15 show up for the flight to Champaign (nobody stops off at a bar and misses the flight!); else, obviously none of the 15 shows up. Let $\mathbf{Y}$ denote the number of nonconnecting passengers who actually show up for the flight. Let H_0 denote the hypothesis that the connecting flight is late, and H_1 the hypothesis that the connecting flight is on time. It is reasonable to assume that the pmf of $\mathbf{Y}$ is the same regardless of which hypothesis is true, and hence we model $\mathbf{Y}$ as a binomial random variable with parameters (90, 0.9). On the other hand, $\mathbf{X}$, the *total* number of passengers showing up for the flight, equals $\mathbf{Y}$ if H_0 is true, while if H_1 is true, then $\mathbf{X} = 15 + \mathbf{Y}$, and thus the pmf of $\mathbf{X}$ *does* depend on which hypothesis is true.

(a) Suppose that the gate agent observes that $\mathbf{X} = 86$. What is $P\{\mathbf{X} = 86\}$ when H_0 is the true hypothesis? What is $P\{\mathbf{X} = 86\}$ when H_1 is the true hypothesis? What is the value of the likelihood ratio when $\mathbf{X} = 86$, and what is the agent's maximum-likelihood decision as to whether the connecting flight is late?

(b) Repeat part (a) for the case when the gate agent observes that $\mathbf{X} = 96$.

(c) The gate agent knows that $\pi_0 = P\{H_0 \text{ is the true hypothesis}\} = 1/3$. For each of the two observations considered in parts (a) and (b), what is the agent's MAP (or Bayesian or minimum-probability-of-error} decision as to whether the connecting flight is late?

(d) What is the probability that all passengers who show up get a seat? Given that all passengers who showed up got a seat, find the (conditional) probability that the connecting flight was late.

2. ["It a'in't about bipartisan politics; it's about …"] The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides *independently* of the other groups whether to support or oppose the motion. *All* members of the group then vote in accordance with the caucus decision.

For those who think that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a bargain price…

(a) Let A, B, C, and D respectively denote the events that the four groups vote for a spending plan that will lead to a balanced budget in seven years. Suppose that the probabilities of these independent events are P(A) = 0.9, P(B) = 0.6 P(C) = 0.5 and P(D) = 0.2. What is the probability that the bill passes?

(b) Let **X** denote the number of votes in favor of the bill. Then, **X** is a discrete random variable. Explain why **X** takes on 16 different values in the range [0,100] and find the pmf of **X**. Compute P{**X** > 50} from the pmf. Is it the same as the answer obtained in part (a)? (Using a spreadsheet/MATLAB/Mathematica will help considerably in doing this part)

(c) The President vetoes the bill. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities P(E) = 0.99, P(F) = 0.4, P(G) = 0.6, and P(H) = 0.1, what is the probability that the motion to override the veto passes ?

(d) Find the pmf of **X**, the number of votes in favor of overriding the veto.

(e) The Clerk of the Senate, being new on the job, has forgotten whether the vote currently being taken is to pass a bill or to override a veto. The Clerk counts the votes and thus knows the value of **X**. Specify the maximum-likelihood decision rule (as to what kind of vote was just taken) in terms of the observed value of **X**. Thus, your answer should be

"If **X**   A then decide that it was a bill, while if **X**   B then decide that it was an override"

where A and B are *disjoint* sets of numbers with |A   B| = 16.

Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

3. ["Party of Five"] A QMR (quintuple modular redundancy) system is a fancier and more expensive version of the TMR system studied in class. It uses 5 identical circuits.

(a) If each circuit has probability p of failing, what is the probability that the majority gate output is incorrect? Ignore the possibility that the majority gate has failed. (Hint: condition on IV and V both failed, one of IV and V failed, and neither IV nor V failed; combine results using the theorem of total probability)

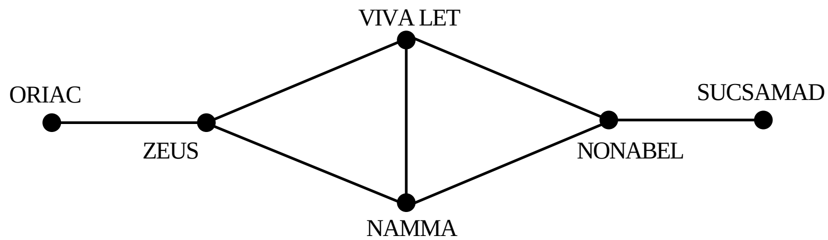
(b) A graph model for the TMR system was discussed in class where it was shown that we must replicate links, e.g. each circuit is represented by more than one link, and if the circuit fails, all these links are removed from the graph. Consider the graph model of the QMR system. If there are no failures, how many paths are there from In to Out? How many links represent each of the circuits?

4. ["Reach out and touch someone"] MiddleEast Bell, a division of NYAAHNYUCKS Corp., has built a telephone network as shown below. Terrorists attack each of the seven links. The attacks may be considered to be independent events, and the attack on a link succeeds in severing the link with probability p. If a link is severed, switches automatically re-route calls so as to avoid the failed link (if possible).

(a) What is the probability of being able to call from ORIAC to SUCSAMAD?

(b) Given that it is possible to call from ORIAC to SUCSAMAD, what is the conditional probability that the ZEUS to NAMMA link is in working condition?

(c) If it is not possible to call from VIVA LET to NAMMA, what is the conditional probability that the ORIAC to ZEUS link is in working condition?



5. Let A denote an event with probability 1/2. Given that A occurred at least 4 times in 10 *independent* trials, what is the probability that it occurred no more than 5 times ?

6. ["Tennis, anyone?"] Consider the following simplified model for a game of tennis. On each serve, let p denote the probability that player A wins the point, and q = 1–p the probability that player B wins the point. Assume that the outcome of each serve is independent of all others. Player A wins the game if the score reaches 4–0, 4–1, or 4–2, while B wins the game if the score reaches 2–4, 1–4, or 0–4. Else, the score reaches 3–3 (called deuce) and from this point onwards, the game continues until one player is two points ahead of the other, and thereby wins the game.

(a) Find the probabilities that the score reaches 4–0, 4–1, or 4–2 and the probabilities that the score reaches 2–4, 1–4, or 0–4. I need 6 answers here!

(b) Find P(score reaches deuce). Show that the sum of the seven probabilities obtained in parts (a) and (b) is 1 regardless of the value of p.

(c) Given that the score is deuce, what is P(A wins the next two points)? (This means A wins the game). What is P(B wins the next two points)? (This means B wins the game). What is the probability that both players win one point each? In this case, the score is tied again, and is also called deuce.

(d) Once the score reaches deuce, there *may* be further deuces until ultimately, either A or B wins both points and thereby wins the game. What is the probability that A ultimately wins the game given that the score is deuce? What is the probability that B ultimately wins the game given that the score is deuce? (Hint: these answers are different from those of part (c)) What is the probability that the game goes on forever with the score continuing to reach deuce after every two points?

(e) Use the results of parts (a)–(d) to express the probability that A wins the game as a function f(p) of p. A little thought shows that B wins with probability f(q) = f(1–p). Now, if p = 0, A wins no points which makes it difficult for him to win any games. Does your function f(p) satisfy f(0) = 0? If not, what does your f(p) give as the probability that A wins a game while losing every point? Similarly, if A wins every point, he is sure to win the game. Does your function f(p) satisfy f(1) = 1? If not, what does your f(p) give as the probability that A loses a game while winning every point?
Other reasonable properties of f(p) are f(0.5) = 0.5, f(p) + f(1–p) =1. Which of these is satisfied by your function f(p)?

(f) Expand f(p) in a Taylor series in the neighborhood of p = 0.5 (only the first two terms are needed) What does this say about the probability of winning a game if $p = 0.5 + \varepsilon$ where ε is very small?

(g) Use your favorite graphing program to sketch f(p) as a function of p for $0 \le p \le 1$. Determine the minimum value of p for which $f(p) \ge 2/3$.