1. Five basketball teams A, B, C, D, and E play in a round-robin tournament, that is, each team plays every other team exactly once.

(a) How many games are played in this tournament?

(b) Mark **one** of the boxes below: The events {A wins the tournament with a 4–0 record} and {C is in last place with a 0–4 record} are

☐ **disjoint** ☐ **independent** ☐ **both** ☐ **neither**

(c) Suppose that the teams playing a game both have probability 1/2 of winning the game. The outcomes of the games are independent of each other. What is the probability that some team wins the tournament with a 4–0 record **and** the last-place team has a 0–4 record?

2. $\mathbf{X}$ and $\mathbf{Y}$ are independent discrete random variables.
$\mathbf{X}$ is a geometric random variable with parameter p, $0 < p < 1$.
$\mathbf{Y} = 1 + \mathbf{Z}$ where $\mathbf{Z}$ is a Poisson random variable with parameter λ where $\lambda > 0$.

(a) TRUE or FALSE? $P\{\mathbf{X} = 2\} < 0.3$ ☐ TRUE ☐ FALSE

$P\{\mathbf{Y} = 2\} < 0.4$ ☐ TRUE ☐ FALSE

(b) For $k \geq 1$, write an expression for $P\{\mathbf{X} = k, \mathbf{Y} = k\}$ in terms of k, p, and λ.

(c) Find the probability that $\mathbf{X}$ equals $\mathbf{Y}$. Express your answer in the simplest possible terms

3. A continuous random variable $\mathbf{X}$ has probability density function (pdf) given by

$$f_{\mathbf{X}}(u) = \exp(-\pi u^2), \quad -\infty < u < \infty.$$

(a) TRUE or FALSE? $\mathrm{var}(\mathbf{X}) = 2\pi$ ☐ TRUE ☐ FALSE

(b) Find the mean and variance of the random variable $\mathbf{Y} = |\mathbf{X}|$.

4. A continuous random variable $\mathbf{X}$ has pdf $f_{\mathbf{X}}(u) = \begin{cases} u \cdot \exp(-u), & u > 0, \\ 0, & u \leq 0. \end{cases}$

What is its hazard rate h(t)?

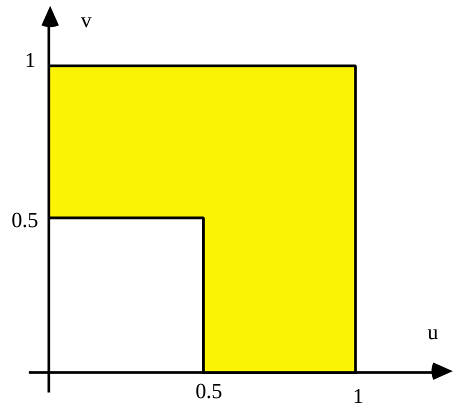
5. $\mathbf{B}$ and $\mathbf{C}$ are random variables with joint pdf

$$f_{\mathbf{B},\mathbf{C}}(u,v) = \begin{cases} u + v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) TRUE or FALSE? $\mathbf{B}$ and $\mathbf{C}$ are independent random variables. ☐ TRUE ☐ FALSE

(b) What is the probability that the roots of the polynomial $x^2 + 2\mathbf{B}x + \mathbf{C}$ are real?

6. The random point $(\mathbf{X}, \mathbf{Y})$ is uniformly distributed on the shaded region shown below.



(a) Let $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$. Find the pdf $f_{\mathbf{Z}}(\alpha)$ of the random variable $\mathbf{Z}$.

Use the "find the CDF and differentiate to find the pdf" method used so often in class. *If* you choose to use the formula $f_{\mathbf{Z}}(\alpha) = \int_{u=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(u, \alpha-u)\,du = \int_{v=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(\alpha-v, v)\,dv$, *you will get* ***no partial credit*** *for an incomplete answer: you will be given (full) credit only if you give a* ***completely correct*** *answer.*

(b) TRUE or FALSE? $E[\mathbf{X}] = E[\mathbf{Y}]$ ☐ TRUE ☐ FALSE

(c) Find the covariance of $\mathbf{X}$ and $\mathbf{Y}$.

7. The jointly Gaussian random variables $\mathbf{X}$ and $\mathbf{Y}$ have means 0 and 7 respectively, variances 4 and 16 respectively, and correlation coefficient 1/16.

(a) Find the probability density function of $\mathbf{Z} = 5\mathbf{X} + \mathbf{Y}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{Z}}(w)$ for all real numbers w.

(b) Find the numerical value of $P\{\mathbf{Y} > 3\mathbf{X}\}$.

SOME USEFUL RESULTS

$\sin(\pi/6) = \cos(\pi/3) = 1/2, \qquad \sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2 \qquad \sin(\pi/3) = \cos(\pi/6) = \sqrt{3}/2$

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1 + x + x^2 + x^3 + \ldots = (1 - x)^{-1} \text{ provided that } |x| < 1$$

$$1 + 2x + 3x^2 + 4x^3 + \ldots = (1 - x)^{-2} \text{ provided that } |x| < 1$$

$$\exp(x) = \sum_{k=0}^{\infty} x^k/k! = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\exp(1) = e = 2.71828\ldots$$

$$\pi = 3.14159\ldots$$

$$\int u\, dv = uv - \int v\, du$$

$$\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1}, & \text{if } n \neq -1, \\ \ln x, & \text{if } n = -1. \end{cases}$$

$$\int \frac{dx}{1+x^2} = \arctan x$$

$$\int \sin x\, dx = -\cos x$$

$$\int \cos x\, dx = \sin x$$

$$\int \exp(ax)\, dx = a^{-1} \exp(ax)$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right]$$

$$\int x\sqrt{a^2-x^2}\, dx = \frac{-1}{3}\left[\sqrt{(a^2 - x^2)^3}\right]$$

$$\int_{-\infty}^{\infty} \exp -(a^2x^2 + 2bx)\, dx = \sqrt{\pi}\, a^{-1} \exp\left(\frac{b^2}{a^2}\right)$$

$\Phi(x)$, the area under the standard Gaussian density function to the left of x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1.(a) $\binom{5}{2} = 10$ games are played.

(b) The events are neither disjoint nor independent.

(c) There are $2^{10} = 1024$ different tournament results possible. The 4–0 winner can be any of 5 teams and the 0–4 loser any of the remaining 4 teams. Hence, there are $5\times 4 = 20$ possible cases where one team is 4–0 and another 0-4. If A and C (say) are 4–0 and 0–4 respectively, then we know the outcomes of the **seven** games AB, AC, AD, AE, BC, DC, EC. The outcomes of other 3 games do not matter. Hence, P{A is 4–0, C is 0–4} $= 2^3/2^{10} = 1/2^7 = 1/128$ and P{*some* team is 4–0 **and** *another* 0–4} = 20/128 = 5/32.

2.(a) TRUE: $P\{\mathbf{X} = 2\} = p(1-p)$ has maximum value 0.25 at $p = 1/2$..

TRUE: $P\{\mathbf{Y} = 2\} = P\{\mathbf{Z} = 1\} = \lambda\cdot\exp(-\lambda)$ has maximum value $\exp(-1) < 0.4$ at $\lambda = 1$.

(b) $P\{\mathbf{X} = k, \mathbf{Y} = k\} = P\{\mathbf{X} = k, \mathbf{Z} = k-1\} = P\{\mathbf{X} = k\}P\{\mathbf{Z} = k-1\}$ by independence.

Hence, $P\{\mathbf{X} = k, \mathbf{Y} = k\} = (1-p)^{k-1}p\cdot\exp(-\lambda)\cdot\lambda^{k-1}/(k-1)!$

(c) $$P\{\mathbf{X} = \mathbf{Y}\} = \sum_{k=1}^{\infty} P\{\mathbf{X} = k, \ \mathbf{Y} = k\} = \sum_{k=1}^{\infty}(1-p)^{k-1}p\cdot\exp(-\lambda)\cdot\lambda^{k-1}/(k-1)!$$

$$= p\cdot\exp(-\lambda)\sum_{k=1}^{\infty}[\lambda(1-p)]^{k-1}/(k-1)! = p\cdot\exp(-\lambda)\cdot\exp(\lambda(1-p)) = p\cdot\exp(-\lambda p).$$

3.(a) FALSE: Actually $\mathbf{X}$ is $N(0,1/2\pi)$.

(b) $$E[\mathbf{Y}] = E[|\mathbf{X}|] = \int_{-\infty}^{\infty}|u|\cdot\exp(-\pi u^2)\,du = 2\int_0^{\infty}u\cdot\exp(-\pi u^2)\,du = -\frac{1}{\pi}\exp(-\pi u^2)\Big|_0^{\infty} = \frac{1}{\pi}.$$

$$E[\mathbf{Y}^2] = E[|\mathbf{X}|^2] = \int_{-\infty}^{\infty}|u|^2\cdot\exp(-\pi u^2)\,du = \int_{-\infty}^{\infty}u^2\cdot\exp(-\pi u^2)\,du = \mathrm{var}(\mathbf{X}) = 1/2\pi.$$

Hence, $\mathrm{var}(\mathbf{Y}) = 1/2\pi - 1/\pi^2 = (\pi-2)/2\pi^2 > 0$.

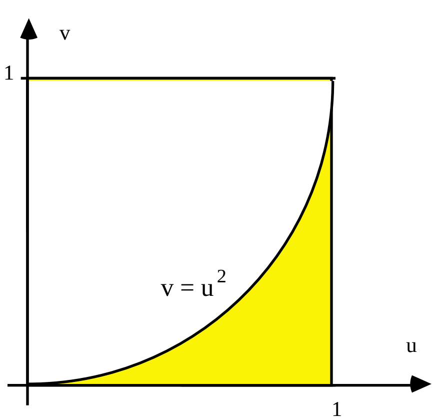
4. $$1 - F_{\mathbf{X}}(t) = \int_t^{\infty}u\cdot\exp(-u)\,du = -u\cdot\exp(-u)\Big|_t^{\infty} + \int_t^{\infty}\exp(-u)\,du = (1+t)\cdot\exp(-t).$$

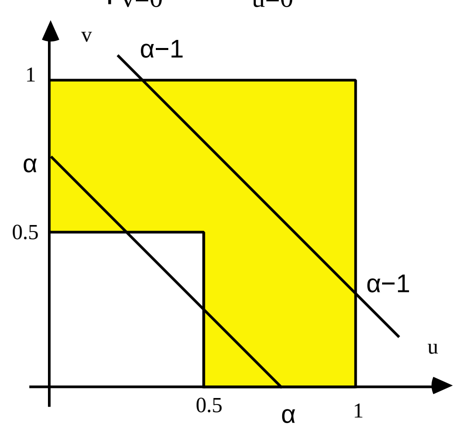
Hence, $h(t) = f(t)/[1 - F_{\mathbf{X}}(t)] = t/(1+t)$ which approaches 1 as $t \to \infty$.

5.(a) FALSE. It is easy to verify that $f_{\mathbf{B}}(u) = (u + 1/2)$ and hence $f_{\mathbf{B},\mathbf{C}}(u,v) \neq f_{\mathbf{B}}(u)f_{\mathbf{C}}(v)$

(b) We wish to find the probability that $\mathbf{B}^2 > \mathbf{C}$. As shown in the diagram below, we have

$$P\{\text{real roots}\} = \int_{u=0}^{1}\int_{v=0}^{u^2} u + v\,dv\,du = \int_{u=0}^{1} uv + v^2/2\Big|_{v=0}^{u^2}du = \int_{u=0}^{1} u^3 + u^4/2\,du = 0.35.$$





Alternatively, $P\{\text{real roots}\} = \int_{v=0}^{1}\int_{u=\sqrt{v}}^{1} u + v \, du\, dv = \int_{v=0}^{1} \left. u^2/2 + uv \right|_{u=\sqrt{v}}^{1} dv$

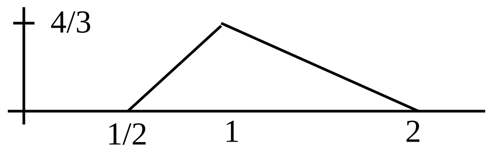
$= \int_{v=0}^{1} (1/2) + v/2 - v^{3/2}\, dv = 0.5 + 0.25 - 0.4 = 0.35.$

6.(a) As discovered on homework, the joint pdf has value 4/3 on the shaded region. $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ takes on values between 1/2 and 2. There are two cases to be considered as shown in the diagram on the previous page.

For $1/2 \le \alpha \le 1$, $F_{\mathbf{Z}}(\alpha) = 2\times(1/2)\times(\alpha-1/2)^2\times(4/3) = (4/3)\times(\alpha-1/2)^2$.

For $1 \le \alpha \le 2$, $1-F_{\mathbf{Z}}(\alpha) = (1/2)\times[1-(\alpha-1)]^2\times(4/3) = (2/3)\times(2-\alpha)^2$.

Hence $f_{\mathbf{Z}}(\alpha) = \begin{cases} (8\alpha - 4)/3, & 1/2 \le \alpha \le 1, \\ (8 - 4\alpha)/3, & 1 \le \alpha \le 2, \\ 0, & \text{elsewhere.} \end{cases}$



(b) TRUE. In fact, the marginal pdfs are also the same (do you see why?)

(c) This problem can be done by brute-force integration. Alternatively, let $\mathbf{A}$ and $\mathbf{B}$ denote independent random variables uniformly distributed on (0,1) so that $(\mathbf{A},\mathbf{B})$ is uniformly distributed on the unit square. Similarly, let $\mathbf{C}$ and $\mathbf{D}$ denote independent random variables uniformly distributed on (0,1/2) so that $(\mathbf{C},\mathbf{D})$ is uniformly distributed on the square with side (1/2). Note that $E[\mathbf{A}] = E[\mathbf{B}] = 1/2$, $E[\mathbf{C}] = E[\mathbf{D}] = 1/4$, and (by independence) $E[\mathbf{AB}] = E[\mathbf{A}]E[\mathbf{B}] = 1/4$ and $E[\mathbf{CD}] = E[\mathbf{C}]\,E[\mathbf{D}] = 1/16$. In what follows, we identify various integrals as the ones arising from calculations involving the above expectations.

Thus, we have $E[\mathbf{X}] = \int_0^1\int_0^1 u\cdot 4/3\, dv\, du - \int_0^{1/2}\int_0^{1/2} u\cdot 4/3\, dv\, du$ (do you see why?)

$= (4/3)\int_0^1\int_0^1 u\, dv\, du - (1/3)\int_0^{1/2}\int_0^{1/2} u\cdot 4\, dv\, du = (4/3)E[\mathbf{A}] - (1/3)E[\mathbf{C}]$

$= (4/3)(1/2) - (1/3)(1/4) = 7/12.$

Similarly, $E[\mathbf{XY}] = \int_0^1\int_0^1 uv\cdot 4/3\, dv\, du - \int_0^{1/2}\int_0^{1/2} uv\cdot 4/3\, dv\, du$

$= (4/3)\int_0^1\int_0^1 u\, v dv\, du - (1/3)\int_0^{1/2}\int_0^{1/2} uv\cdot 4\, dv\, du = (4/3)E[\mathbf{AB}] - (1/3)E[\mathbf{CD}]$

$= (4/3)(1/4) - (1/3)(1/16) = (1/3)(1 - 1/16) = 5/16.$ Finally, we get

$\text{cov}(\mathbf{X},\mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = 5/16 - (7/12)^2 = (45-49)/144 = -4/144 = -1/36.$

7.(a) $\mathbf{Z} = 5\mathbf{X} + \mathbf{Y}$ is a Gaussian random variable with mean

$E[\mathbf{Z}] = E[5\mathbf{X} + \mathbf{Y}] = 5\times E[\mathbf{X}] + E[\mathbf{Y}] = 5\times 0 + 7 = 7$

and variance $\text{var}(\mathbf{Z}) = 5^2\text{var}(\mathbf{X}) + 1^2\text{var}(\mathbf{Y}) + 2\times5\times1\times\text{cov}(\mathbf{X},\mathbf{Y})$

$= 25\times4 + 16 + 10\rho\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = 100 + 16 + 10\times(1/16)\times2\times4 = 121 = 11^2$. Hence,

$f_{\mathbf{Z}}(w) = 1/(11\sqrt{2\pi})\cdot\exp(-(w-7)^2/242),\ -\infty < w < \infty.$

(b) $P\{\mathbf{Y} > 3\mathbf{X}\} = P\{3\mathbf{X} - \mathbf{Y} < 0\}$. But $3\mathbf{X} - \mathbf{Y}$ is a Gaussian random variable with mean $E[3\mathbf{X} - \mathbf{Y}] = 3\times E[\mathbf{X}] - E[\mathbf{Y}] = -7$ and

variance $3^2\text{var}(\mathbf{X}) + (-1)^2\text{var}(\mathbf{Y}) + 2\times3\times(-1)\times\text{cov}(\mathbf{X},\mathbf{Y})$

$=9\times4 + 16 - 6\rho\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = 36 + 16 - 6\times(1/16)\times2\times4 = 49 = 7^2.$

Hence, $3\mathbf{X} - \mathbf{Y}$ is $N(-7,7^2)$ and thus $P\{3\mathbf{X} - \mathbf{Y} < 0\} = \Phi\left(\frac{0-(-7)}{7}\right) = \Phi(1) = 0.8413.$