University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313
Final Examination

Friday, May 8, 1998
1:30 p.m. — 4:30 p.m.
Room 124 Burill Hall

Name ______________________________

University ID Number ______________________________

Signature ______________________________

Section:

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INSTRUCTIONS

This exam is closed book and closed notes, except that three $8.5'' \times 11''$ sheets of notes (both sides) are allowed. Except for pocket calculators distributed in class, other types of calculators, laptop computers, tables of integrals, etc., may not be used.

The exam consists of 9 problems worth a total of 300 points. Of these, 75 are **extra-credit points**. That is, a score of only 225 points is needed to receive the full 45% credit given to the final exam. Scores in excess of 225 points will compensate for Hour Exam and Homework scores. Notice that no partial credit will be given for the extra-credit problems.

Write your answers in the spaces provided. Show *all* your work, except in true/false and multiple choice problems. If you need extra space, use the back of the previous page. Partial credit will only be given for substantial progress on a problem.

Grading

1.	48 points	______
2.	42 points	______
3.	15 points	______
4.	25 points	______
5.	20 points	______
6.	25 points	______
7.	25 points	______
8.	25 points	______
extra:	25 points	______
extra 9.	50 points	______
	TOTAL	______

Problem 1 (48 points)

Check the appropriate box for each of the statements below. **No justification is required.** *However, so as not to give points for random guessing, you will receive +3 points for correct answer, 0 points for no answer, and −3 points for wrong answer.*

a. The following are statements about an arbitrary continuous random variable X with finite mean and variance. Answer True if the statement is true for **all** such random variables, otherwise answer False.

True	False	
☐	☐	If $\mathrm{Var}(X) = 0$ then $E[X] = 0$.
☐	☐	$(E[X])^2 \leq E[X^2]$
☐	☐	$g(X)$ is a continuous random variable for all functions $g(\cdot)$
☐	☐	If $Y = \lvert X\rvert$, then $F_Y(v) = F_X(v) + F_X(-v)$ for all $v > 0$
☐	☐	If $Y = \lvert X\rvert$, then $f_Y(v) = f_X(v) + f_X(-v)$ for all $v > 0$

b. The following are statements about arbitrary jointly continuous random variables X, Y with joint probability density function $f_{X,Y}(u, v)$. Answer True if the statement is true for **all** such random variables, otherwise answer False.

True	False	
☐	☐	$P\{X + Y \geq 1\} = \int_{-\infty}^{\infty} \int_{1-u}^{\infty} f_{X,Y}(u, v)\, dv du$
☐	☐	$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u f_{X,Y}(u, v)\, dv du$
☐	☐	$f_{X+Y} = f_X * f_Y$, where "$*$" denotes convolution

c. The following are statements about arbitrary random variables X and Y, with finite mean and variance. Answer True if the statement is true for **all** such random variables, otherwise answer False.

True	False	
☐	☐	If X,Y are uncorrelated then $E[X^2Y^2] = E[X^2]E[Y^2]$
☐	☐	If X,Y are i.i.d. then $\mathrm{Var}(X - Y) = 0$
☐	☐	If X,Y have equal variances, then $X + Y$, $X - Y$ are uncorrelated
☐	☐	If $X + Y$ and $X - Y$ are uncorrelated then $\mathrm{Var}(X) = \mathrm{Var}(Y)$

d. The following are statements about independent, identically distributed, random variables $X_1, X_2, \ldots, X_n$. The random variable Y is defined by:

$$Y = X_1 + X_2 + \cdots + X_n$$

Assume that the random variables $X_1, X_2, \ldots, X_n$ have finite mean μ and finite variance σ^2, but are arbitrary otherwise. Answer True if the statement is true for **all** such random variables, otherwise answer False.

True	False	
☐	☐	If $\widehat{X} = Y/n$, then $\mathrm{Var}(\widehat{X}) = \sigma^2/n$
☐	☐	Y is a Gaussian random variable
☐	☐	If $X_1, \ldots, X_n$ are Gaussian, then $P\{Y \le n\} = P\{X_1 \le 1\}$
☐	☐	If $\mu = \sigma = 1$, then the variance of the product $X_1 \cdot X_2 \cdots X_n$ is $2^n - 1$

Check the appropriate box for each of the statements above. **No justification is required**. *However, so as not to give points for random guessing, you will receive +3 points for correct answer, 0 points for no answer, and −3 points for wrong answer.*

Problem 2 (42 points)

The following are multiple choice questions: check a **single box** *for each of these questions.* **No justification is required**. *However, so as not to give points for random guessing, you will receive* $+6$ *points for correct answer,* 0 *points for no answer, and* -2 *points for wrong answer.*

1. There are n fortune cookies in a jar, and one of them contains a prize. A line of n people is being formed. Each person in the line, in his or her turn, draws a cookie from the jar, opens it to see if it contains the prize, and then eats it. Where should you stand in this line to maximize your chance of drawing the prize cookie?

- ☐ first position in line
- ☐ second position in line
- ☐ last position in line
- ☐ it does not matter where you stand

2. A communication system transmits a signal $\mathcal{S}$ over a noisy channel. The transmitted signal may have one of two amplitudes $H_0 : \mathcal{S} = -2$ or $H_1 : \mathcal{S} = +4$. The *a priori* probabilities of the two hypotheses are $P(H_0) = 0.1$ and $P(H_1) = 0.9$. The received signal $\mathcal{R}$ is given by $\mathcal{R} = \mathcal{S} + X$, where $X \sim \mathcal{N}(0, 1)$ is a unit Gaussian random variable. Which of the following represents the ***maximum-likelihood*** decision rule for this system?

- ☐ decide H_0 if $\mathcal{R} < 0$; decide H_1 if $\mathcal{R} \geq 0$
- ☐ decide H_0 if $\mathcal{R} < 1$; decide H_1 if $\mathcal{R} \geq 1$
- ☐ always decide H_1
- ☐ none of the above

3. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. Then the random variable $\mathcal{Z} = X_1 + X_2$ is a Poisson random variable with parameter:

- ☐ $\max(\lambda_1, \lambda_2)$
- ☐ $\min(\lambda_1, \lambda_2)$
- ☐ $\lambda_1 + \lambda_2$
- ☐ $\lambda_1\lambda_2/(\lambda_1 + \lambda_2)$

4. Let X, Y be ***independent*** jointly continuous random variables. Assume that it is known that $f_X(u) = f_X(-u)$ and $f_Y(v) = f_Y(-v)$ for all real numbers u and v. Then:

- ☐ $P\{X + Y \geq 0\} = 1$
- ☐ $P\{X + Y \geq 0\} = 0.5$
- ☐ $P\{X + Y \geq 0\} = P\{X \geq 0\} \cdot P\{Y \geq 0\}$
- ☐ $P\{X + Y \geq 0\}$ cannot be computed from the given data

5. Let X and Y denote the number of heads and the number of tails, respectively, observed on 10 tosses of a fair coin. Further let $\mathcal{Z} = X - Y$. Then

- ☐ $\mathsf{Cov}(X, \mathcal{Z}) = 0.25$
- ☐ $\mathsf{Cov}(X, \mathcal{Z}) = -2.5$
- ☐ $\mathsf{Cov}(X, \mathcal{Z}) = 5$
- ☐ $\mathsf{Cov}(X, \mathcal{Z}) = 2.5$

6. The random variables X and Y are jointly Gaussian. It is also known that $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$, so that X and Y are identically distributed. Define two new random variables $\mathcal{W} = 2X + Y$ and $\mathcal{Z} = X - 2Y$, and consider the following three statements:

(i) The random variables $\mathcal{W}$ and $\mathcal{Z}$ are jointly Gaussian.

(ii) The random variables $\mathcal{W}$ and $\mathcal{Z}$ are independent.

(iii) The random variables $\mathcal{W}$ and $\mathcal{Z}$ are identically distributed.

Which of these three statements are true for *all* such random variables X and Y?

- ☐ only (i) is true
- ☐ only (iii) is true
- ☐ only (i) and (iii) are true
- ☐ all of (i), (ii), and (iii) are true

7. Suppose that a fair coin is tossed n times, and let the random variable X_n denote the number of heads observed on the n tosses. Consider the following statement: There exists some value of n such that $P\{\, 0.49n \le X_n \le 0.51n \,\} \ge 0.999$.

- ☐ This statement follows from the WLLN, but it does not follow from the CLT
- ☐ This statement follows from the CLT, but it does not follow from the WLLN
- ☐ This statement follows from the WLLN, it also follows from the CLT
- ☐ This statement does not follow from the WLLN; it also doesn't follow from the CLT

In these questions we have use the following terminology: WLLN stands for the Weak Law of Large Numbers, while CLT stands for the Central Limit Theorem.

Problem 3 (15 points)

A consumer electronics company buys 20%, 30%, and 50% of their resistors from Supplier A, Supplier B, and Supplier C, respectively. It is known that 0.5%, 1.0%, and 0.25% of the resistors produced by Supplier A, Supplier B, and Supplier C, respectively, are defective.

a. What percentage of the total number of resistors purchased by this consumer electronics company are defective?

$\boxed{P\{D\} =}$

b. If a resistor purchased by the company is defective, what is the probability that it came from Supplier A?

$\boxed{P\{A|D\} =}$

Problem 4 (25 points)

Consider independent trials of an experiment which consists of flipping a fair coin. Let the random variable X denote the waiting time until the first agreement of two ***successive*** flips. That is, X is the number of trials required to observe either HH or TT for the first time. This number ***includes*** the (last) two trials that result in HH or TT.

a. What is $P\{X = 5\}$?

$P\{X = 5\} =$

b. Compute $P\{X = k\}$ for all integer $k = 0, 1, 2, \ldots$.

$P\{X = k\} = \begin{cases} \end{cases}$

c. Find the expected value of X.

$$\boxed{E[X] \ =}$$

Problem 5 (20 points)

Your boss tells you to model a certain parameter as a continuous random variable X, with probability density function:

$$f_X(u) \;=\; e^{\mathbf{c}u^2} \qquad \text{for} \;\; -\infty < u < +\infty.$$

Find the value(s) of the constant **c** for which this is a valid probability density function.

☐ $f_X(u)$ is a valid probability density function for $\boxed{\mathbf{c} = \qquad\qquad}$

☐ $f_X(u)$ is not a valid probability density function for any value of **c**.

Problem 6 (25 points)

The Bernoulli factory produces disk drives. The lifetime of a disk drive produced by the factory is an continuous random variable X whose probability density function is given by

$$f_X(u) \;=\; \begin{cases} e^{-u}\,, & u > 0 \\ 0\,, & \text{otherwise} \end{cases}$$

Next week, a new machine will be installed at the Bernoulli factory. As a result, the lifetime of a disk drive produced by the factory will be given by $Y = \sqrt{X}$.

a. Find the mean and the variance of X.

$E[X] \;=$

$\mathrm{Var}(X) \;=$

b. Find the probability density function of Y.

$$f_Y(v) \;=\; \begin{cases} \\ \\ \end{cases}$$

Problem 7 (25 points)

Alice and Bob would like to meet each other at the Random House Cafe. Due to heavy traffic in Random City, Alice arrives at the Random House Cafe at time X that is uniformly distributed between 1:00pm and 2:00pm. Bob arrives at the cafe at time Y that is also uniformly distributed between 1:00pm and 2:00pm. The random variables X and Y are ***independent***.

If Bob is not there when Alice arrives at the cafe, Alice will wait for Bob for **30 minutes** at most, and then leave. If Alice is not there when Bob arrives at the cafe, Bob will wait for Alice for **15 minutes** and then leave. Let $\mathcal{E}$ denote the event that Alice and Bob meet.

a. Describe the event $\mathcal{E}$ in terms of event(s) associated with the random variables X and Y.

$\mathcal{E} =$

b. Find the probability that Alice and Bob meet.

$P\{\mathcal{E}\} =$

Problem 8 (25 points)

A graph $\mathcal{G}$ on a 25 vertices, labeled $1, 2, \ldots, 25$, is created as follows. For each pair of distinct vertices, we flip a biased coin that lands on heads with probability p. If the coin lands on heads, we draw an edge between the two vertices, otherwise we leave the two vertices unconnected. You may assume that all the $\binom{25}{2} = 300$ coin flips are independent trials.

a. Let $X_{i,j}$ be a discrete random variable defined as follows: $X_{i,j} = 1$ if there is an edge between vertices i and j, and $X_{i,j} = 0$ otherwise.

(a) Find the mean and variance of $Y = X_{1,2} + X_{2,3}$.

$E[Y]$ =	Var(Y) =

(b) Let $Y_1 = X_{1,3} + X_{3,4}$ and $Y_2 = X_{1,2} + X_{4,5}$. Are the random variables Y and Y_1 independent? Are the random variables Y and Y_2 independent?

- ☐ The random variables Y and Y_1 are independent
- ☐ The random variables Y and Y_1 are **not** independent

- ☐ The random variables Y and Y_2 are independent
- ☐ The random variables Y and Y_2 are **not** independent

You must check the appropriate box in each case **and** explain your answer.

The *degree* of a vertex i, denoted D_i, is the number of edges that have this vertex as one of its endpoints. In other words, we have:

$$D_i = \sum_{\substack{j=1 \\ j \neq i}}^{25} X_{i,j}$$

for $i = 1, 2, \ldots, 25$. Thus D_i is a discrete random variable that takes on values $0, 1, \ldots, 24$.

b. Find the probability mass function of D_1.

$P\{D_1 = k\} =$

c. Find the mean and the variance of D_1.

$E[D_1] =$

$\mathrm{Var}(D_1) =$

Extra-credit (25 more points)

d. Compute the covariance of the random variables D_1 and D_2.

$\boxed{\mathsf{Cov}(D_1, D_2) \; = \qquad\qquad}$

Extra-credit: Problem 9 (50 points)

Construct random variables X and Y, by specifying either the joint probability mass function of X, Y or the joint probability density function of X, Y, such that the following two conditions:

a. $P\{X < Y\} \neq P\{Y < X\}$

b. X and Y are identically distributed

are **both** satisfied. To receive credit you must specify the joint distribution of X and Y *precisely*, and also *prove* that the two conditions (**a**) and (**b**) are satisfied for this joint distribution.