University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313
Final Examination

Monday, December 14, 1998
8:00 a.m. — 11:00 a.m.
Rooms 245, 260 Everitt Labs

Name ____________________________

University ID Number ____________________________

Signature ____________________________

Section:
- ☐ B 9 MWF
- ☐ D 11 MWF

INSTRUCTIONS

This exam is closed book and closed notes, except that three $8.5'' \times 11''$ sheets of notes (both sides) are allowed. Except for pocket calculators distributed in class, other types of calculators, laptop computers, tables of integrals, etc., may not be used.

The exam consists of 6 problems worth a total of 275 points. Of these, 50 are **extra-credit points**. That is, a score of only 225 points is needed to receive the full 45% credit given to the final exam. Scores in excess of 225 points will compensate for Hour Exam and Homework scores. Note that **no partial credit** will be given for the extra-credit problems.

Write your answers in the spaces provided. Show *all* your work, except in multiple choice problems. If you need extra space, use the back of the previous page. Partial credit will only be given for substantial progress on a problem.

Grading

1.	90 points	______
2.	39 points	______
3.	24 points	______
extra:	25 points	______
4.	42 points	______
5.	30 points	______
extra 6.	25 points	______
	TOTAL	______

Problem 1 (96 points)

The following are multiple choice questions: check a **single box** *for each of these questions.* **No justification is required**. *However, so as not to give points for random guessing, you will receive* $+9$ *points for correct answer,* 0 *points for no answer, and* -3 *points for wrong answer.*

1. The events A and B in the sample space Ω are arbitrary, with probabilities $P(A)$ and $P(B)$ such that $0 < P(A) < 1,\ \ 0 < P(B) < 1$.

Define two random variables $X = \mathcal{I}_A(\omega)$ and $Y = \mathcal{I}_B(\omega)$, where $\mathcal{I}_E(\omega)$ is the *indicator* function of the set E, i.e., $\mathcal{I}_E(\omega) = 1,\ \omega \in E$, and $\mathcal{I}_E(\omega) = 0,\ \omega \notin E$. Which of the following four statements are true?

(i) X and Y are discrete random variables

(ii) $P(X = 0, Y = 0) = P(A^c B^c)$

(iii) If $A \subset B$ then $X(\omega) \le Y(\omega)$ for all $\omega \in \Omega$

(iv) If $A \subset B$ then $F_X(u) \le F_Y(u)$ for all $u \in \mathbb{R}$

☐ Only (i) is true
☐ Only (i) and (ii) are true
☐ Only (i), (ii) and (iii) are true
☐ All the above four are true

2. In this problem X denotes an **arbitrary continuous** random variable with finite mean and variance. As usual, $F_X(u)$ is the CDF of X, and $f_X(u)$ is the pdf of X.

(a) Which of the following four statements are properties of $F_X(u)$?

(i) $\lim_{u\to-\infty} F_X(u) = 0$

(ii) If $F_X(a) < F_X(b)$ then $a < b$

(iii) $P(|X| < 1) = 2F_X(1) - 1$

(iv) $F_X(u) = 1/2$ for some $u \in (-\infty, +\infty)$

☐ Only (i) and (ii) are properties of $F_X(u)$
☐ Only (i), (ii) and (iii) are properties of $F_X(u)$
☐ Only (ii), (iii) and (iv) are properties of $F_X(u)$
☐ All four are properties of $F_X(u)$

(b) Which of the following four statements about X are true?

(i) $\mathrm{Var}(X - E[X]) = \mathrm{Var}(X)$

(ii) $(E[X])^2 \le E[X^2]$

(iii) $P(a \le X \le b) = P(e^a \le e^X \le e^b),\ \forall a, b \in \mathbb{R}$

(iv) $P(1/X \le a) = 1 - P(X \le 1/a),\ \forall a \in \mathbb{R}$

☐ Only (ii) and (iii) are true
☐ Only (iii) and (iv) are true
☐ Only (i), (ii) and (iii) are true
☐ All four are true

3. The following are statements about arbitrary jointly continuous random variables X, Y with joint pdf $f_{X,Y}(u, v)$. As usual, $f_X(u)$ and $f_Y(v)$ represent the marginals of X and Y, respectively. In addition, X and Y have finite means and variances. Check only **one** box for parts (a) and (b).

(a) Which of the following four statements is true for *all* such random variables X and Y?

- ☐ $f_{X+Y} = f_X * f_Y$, where "$*$" denotes convolution
- ☐ $E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v f_{X,Y}(u, v)\, dudv$
- ☐ If X, Y are uncorrelated, then X, Y are independent
- ☐ If X, Y are uncorrelated, then X^2, Y^2 are uncorrelated

(b) If it is known that X and Y defined above have **identical variances**, which of the following four statements is true?

- ☐ If X, Y are uncorrelated, then $\mathrm{Var}(X - Y) = 0$
- ☐ $X + Y$ and $X - Y$ are correlated random variables
- ☐ $\mathrm{Var}(2X - Y) = \mathrm{Var}(X - 2Y)$
- ☐ $E[X^2] = E[Y^2]$

4. Let X be a continuous random variable with pdf $f_X(u)$ that is an even function of u, i.e., $f_X(u) = f_X(-u)$ for all u, $-\infty < u < \infty$. Let $F_X(u)$ denote its CDF, and let $\mathrm{Var}(X) = 4$. Define another random variable $Y = |X|$. Answer the following three questions based on the above facts.

(a) Which of the following four statements are true about X and Y?

(i) $E[X] = 0$ (iii) $E[Y^2] = 4$

(ii) $E[Y] = 2$ (iv) X and Y are uncorrelated random variables

- ☐ Only (i) and (ii) are true
- ☐ Only (i) and (iii) are true
- ☐ Only (iii) and (iv) are true
- ☐ Only (i), (iii) and (iv) are true

(b) Which of the following three statements are true about $F_X(u)$?

(i) $P(X > u) = F_X(-u)$ for all $u \in \mathbb{R}$

(ii) $F_X(u) = F_X(-u)$ for all $u \in \mathbb{R}$

(iii) $P(X^2 + 3X + 2 < 0) = P(X^2 - 3X + 2 < 0)$

- ☐ Only (i) is true
- ☐ Only (i) and (iii) are true
- ☐ Only (ii) and (iii) are true
- ☐ All of (i), (ii) and (iii) are true

5. The number of different 5-letter words that can be formed from the letters "COCOA" is

- ☐ 10
- ☐ 30
- ☐ 60
- ☐ 120

6. Let the random variable N denote the number of tosses of a fair coin until the sequence HT or TH occurs for the first time. Then $P(N = 4) =$

- ☐ $1/16$
- ☐ $1/8$
- ☐ $1/4$
- ☐ $3/16$

7. Let X and Y denote the number of 1's and 6's, respectively, observed on 18 rolls of a fair die.

- ☐ $\mathsf{Cov}(X, Y) = 0$
- ☐ $\mathsf{Cov}(X, Y) = -0.5$
- ☐ $\mathsf{Cov}(X, Y) = -3$
- ☐ $\mathsf{Cov}(X, Y) = 0.5$

Check the appropriate box for each of the statements above. **No justification is required**. *However, so as not to give points for random guessing, you will receive* $+9$ *points for correct answer,* 0 *points for no answer, and* -3 *points for wrong answer.*

Problem 2 (39 points)

"Homerun" Calvin, as they call him, has a probability of p of hitting a homerun every time he steps up to the plate. This season he came up to bat 500 times, and you can assume that his performance each time is independent of his other performances.

a. What is the probability that he hit at least 70 homeruns this season?

$P(\text{At least } 70 \text{ homeruns}) =$

b. Now suppose $p = 0.13$ (pretty impressive, huh?). Calculate a numerical value to your answer in part (a).
Hint: Direct evaluation is way too long. So use a Gaussian approximation to the actual pmf. That is, approximate the sum of 500 Bernoulli ("one-zero") random variables by an appropriate Gaussian random variable.

$P(\text{At least } 70 \text{ homeruns}) =$

c. “Homerun” Calvin’s philosophy is “if I can do it once, I can do it again”.
He says he is 90% sure of hitting at least 70 homeruns next season too. How many times, n, would he actually have to bat to have this statement come true, i.e., how many times does he have to bat to have a 90% chance of hitting at least 70 homeruns? Again, use the Gaussian approximation to the actual pmf. Recall that the probability of him hitting a homerun at bat is 0.13.

$$\boxed{n = \left\{ \quad \right.}$$

Problem 3 (24 points)

The "Coco Cabeza" cookie factory manufactures chocolate chip cookies. Boxes of cookies are fed onto conveyor belt A toward a check point. The probability that exactly k boxes on belt A pass the check point in one minute is given by the Poisson pmf $p_A(k) = \dfrac{e^{-3}\,3^k}{k!}$, $k = 0, 1, 2, \ldots$

a. vWhat is the probability of the event E, that exactly 4 boxes go past the check point in one minute, **and** exactly 6 go past in the next minute?

$P(E) = P\{4 \text{ in first minute and } 6 \text{ in second}\} =$

b. Realizing that one conveyor belt is not enough to handle their cookies, Coco Cabeza builds another conveyor belt, B. The probability that exactly k boxes on belt B pass the check point in one minute is given by the Poisson pmf $p_B(k) = \dfrac{e^{-6}\,6^k}{k!}$, $k = 0, 1, 2, \ldots$.

Calvin-the-cookie-inspector visits the factory for a random inspection. He randomly chooses one of the conveyor belts and observes that exactly 4 boxes go past in the first minute and 6 in the next minute, i.e., event E occurs. What is the conditional probability that he chose conveyor belt A?

$P\{\text{chose} A | E\} =$

Extra-credit (25 more points)

c. Calvin "suspects" that there is something wrong with the cookies on belt A, and so he removes **every second** box off that belt. Unknown to Calvin, his apprentice Hobbes takes away **every second** cookie box that Calvin removes, for further "inspection". If $\mathcal{N}$ is a random variable denoting the number of boxes that Hobbes tucks away in one minute, find the pmf of $\mathcal{N}$, i.e., find $P(\mathcal{N} = k),\ k = 0, 1, 2, \ldots$

$$\boxed{P\{\mathcal{N} = k\} \ = \ \left\{ \qquad \qquad \qquad \qquad \right.}$$

Problem 4 (42 points)

Random variables X and Y are jointly Gaussian with joint density given by

$$f_{X,Y}(u,v) = \frac{1}{\pi\sqrt{3}} \exp\left\{-\frac{2}{3}\left[u^2 + v^2 - uv\right]\right\}, \; -\infty < u, v < \infty$$

a. If a new random variable Z is defined by $Z = X - Y$, find the pdf of Z.

$$\boxed{f_Z(u) \; = \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.}$$

b. Calculate $P(X > Y)$.

$$\boxed{P(X > Y) \; =}$$

c. Now define two random variables $Z = 2X + Y + 1$ and $W = Y - 2X + 1$. Find the joint probability density function of Z and W.

$$f_{Z,W}(a,b) = \begin{cases} \\ \\ \end{cases}$$

Are the random variables Z and W independent?

- ☐ The random variables Z and W are independent
- ☐ The random variables Z and W are **not** independent

You must check the appropriate box **and** explain your answer.

Problem 5 (30 points)

Suzie wants Calvin to play "Doctor-Doctor" with her, much to Calvin's disgust. To decide if Calvin should play or not, they play this game: Suzie picks a real number uniformly distributed between 1 and 3, and Calvin independently picks a real number uniformly distributed between 1 and 6. They both tell Hobbes their numbers. If Calvin's number falls between Suzie's number and twice her number, he loses.

a. Find the probability that Calvin loses.

$P(\text{Calvin loses}) =$

b. They decide to play a "best-of-three" set, i.e., the first person to win 2 games out of three wins. If the random variable $\mathcal{N}$ denotes the number of games played before someone wins, find the pmf of $\mathcal{N}$.

$$p_{\mathcal{N}}(k) \; = \; \begin{cases} \\ \\ \end{cases}$$

c. Find the expected number of games played, $E[\mathcal{N}]$.

$$E[\mathcal{N}] \; =$$

Extra-credit: Problem 6 (25 points)

Calvin and Suzie (yet again!) are playing a board game with a fair four-faced die, with faces labelled $1, 2, 3$, and 4. On their turn, each player rolls the die and moves the number of spaces on the die. If a 1 or a 2 is rolled, the player can roll again. During one of Calvin's turns Suzie has to leave the room for a while. When she comes back she sees that Calvin has moved forward 10 spaces. He insists that he rolled **only** 1's and 2's. What is the probability that he is lying, i.e., that he rolled 3's or 4's (or both) also?

Generalize this to Calvin having moved n spaces.

$P(\text{Calvin is lying}) =$