University of Illinois Spring 1998

ECE 313: Problem Set #14

Assigned: May 1, 1996
Due: May 6, 1996
Reading: Ross, Chapter 7, sections 1–3, Chapter 8, sections 1–4;

1. Suppose that n fair dice are rolled. Let X, Y denote, respectively, the number of dice that show 1's and the number of dice that show 2's. Find $\mathsf{Cov}(X, Y)$.

2. Let $E[X] = 1$, $E[Y] = 4$, $\mathsf{Var}(X) = 4$, $\mathsf{Var}(Y) = 9$, and $\rho_{X,Y} = 0.1$.

(**a**) If $Z = 2(X+Y)(X-Y)$, find $E[Z]$.

(**b**) If $T = 2X + Y$ and $U = 2X - Y$, find $\mathsf{Cov}(T, U)$?

(**c**) If $W = 3X + Y + 2$, find $E[W]$ and $\mathsf{Var}(W)$.

(**d**) If W is as above, and X, Y are jointly Gaussian random variables, what is $P\{W > 0\}$?

3. Let $X_1, X_2, \cdots$ be an infinite sequence of independent, identically distributed, random variables with mean μ and variance σ^2. We define $Y_n = X_n + X_{n+1} + X_{n+2}$, for $n = 1, 2, \cdots$. For each $k \geq 0$, compute $\mathsf{Cov}(Y_n, Y_{n+k})$.

4. Suppose that X and Y are jointly Gaussian random variables. It is further known that:

$$E[X] = 0, \qquad E[Y] = 0, \qquad \mathsf{Var}(X) = \sigma_1^2, \qquad \mathsf{Var}(Y) = \sigma_2^2, \qquad \rho(X, Y) = \rho$$

Find an angle θ such that $Z = X\cos\theta + Y\sin\theta$ and $W = Y\cos\theta - X\sin\theta$ are *independent* Gaussian random variables. You may express your answer in terms of a trigonometric function of σ_1, σ_2 and ρ. In particular, what is the value of θ if $\sigma_1 = \sigma_2$?

5. A continuous random variable X has the following probability density function

$$f_X(u) = \begin{cases} 24u^{-4} & u \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

For $\delta > 0$, define the function $q(\delta) = P\{|X - \mu| > \delta\}$, where $\mu = E[X]$.

(**a**) Give an expression for the actual value of $q(\delta)$ in terms of δ.

(**b**) Use Chebyshev inequality to obtain an upper bound on $q(\delta)$.

(**c**) Compute the values of $q(\delta)$ and the upper bound from (**b**) for $\delta = 0.5, 1, 2, 3, 5$. Use this data to sketch a graph of $q(\delta)$ and the upper bound thereupon.

6. In this problem let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables that are not necessarily identically distributed. However, assume that

i. All the random variables in the sequence have the same finite mean μ.
ii. There exists a number $M_1 > 0$, such that $\mathrm{Var}(X_i) \leq M_1$ for all i.
iii. There exists a number M_2, such that $\mathrm{Cov}(X_i, X_j) \leq M_2$ for all i and j with $i \neq j$.

Show that the weak law of large numbers (WLLN) holds in each of the following cases

(**a**) The random variables $X_1, \ldots, X_n, \ldots$ are independent.

(**b**) For $k \geq 2$ each random variable X_k depends on X_{k-1} and X_{k+1}, but is independent of all the other random variables in the sequence. Assume that X_1 depends only on X_2.

(**c**) The random variables are *not* independent, but it is known that $M_2 < 0$.

7. The Sirrah Poll wishes to assess the popularity of Bill Clinton following the recent midterm (not the exam!) elections. To this end, a random sample of n voters is asked for opinions. The opinion of the i-th voter is coded as X_i, where $X_i = 1$ if the voter supports the president and $X_i = 0$ otherwise. The Sirrah pollsters treat X_is as independent random variables with $P\{X_i = 1\} = p$ for all i, and estimate p as the sample average $\widehat{p} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$. To be fairly sure of this estimate, the Sirrah Poll wants the following inequality to hold

$$P\{\,|\widehat{p} - p| \geq 0.02\,\} \;\leq\; 0.05$$

This would allow the media to announce that the Sirrah Poll has found that $100\widehat{p}\,\%$ of the voters support the president, and that the margin of error of the poll is $\pm 2\%$.

(a) Suppose that $p = 0.3$. Use the WLLN to find the minimum number N of voters that need to be surveyed to guarantee that the above inequality holds.

(b) The Sirrah Poll naturally wants to minimize the number of voters surveyed in order to cut down the costs. However, the pollsters do not know the value of p (they wouldn't need the poll if they did!). How many voters should the Poll survey, so that the above inequality would be satisfied regardless of the value of p?

8. Let $X_1, X_2, , \ldots$ be a sequence of independent random variables with mean $E[X_i] = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$. Define

$$S_n \;=\; \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}}$$

Compute and compare the results given by Chebyshev's inequality and the CLT for the probabilities $P\{-k \leq S_n \leq k\}$ for $k = 1, 2, 3$. Assume that n is large.

9. Each of the 300 workers of a factory takes his lunch at random in one of the three competing restaurants. Approximately how many seats should each restaurant have so that, on the average, at most one in 20 workers will remain unseated?