University of Illinois

Problem Set 13

ECE 313
Spring 1998

Assigned : Friday, April 24, 1998
Due : Monday, May 4, 1998
Reading : Ross, Chapters 6,7

1. Three points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are independently selected at random from the interval $[0, 2]$.

   (a) What is the probability that $\mathbf{X}_2$ lies between $\mathbf{X}_1$ and $\mathbf{X}_3$?

   (b) What is the probability that the largest of the three is greater than 1?

   (c) What is the probability that the largest of the three is greater than the sum of the other two?

2. Suppose $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are independent random variables that are each equally likely to be either 1 or 2. Find the pmf of

   (a) $\mathbf{XYZ}$

   (b) $\mathbf{XY} + \mathbf{YZ} + \mathbf{XZ}$

   (c) $\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$

3. If $\mathbf{U}$ is uniform on $(0, 2\pi)$ and $\mathbf{Z}$, independent of $\mathbf{U}$, is exponential with parameter $\lambda = 1$,

   (a) Find the joint density of the random variables $\mathbf{X}, \mathbf{Y}$ defined by

   $$\mathbf{X} = \sqrt{2\mathbf{Z}} \cos \mathbf{U}$$
   $$\mathbf{Y} = \sqrt{2\mathbf{Z}} \sin \mathbf{U}$$

   (b) Show that $\mathbf{X}$ and $\mathbf{Y}$ are independent unit normal RVs, i.e., that they are independent and each of them is distributed as $N(0, 1)$.

   (c) What is the pdf of the random variable $\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$?

4. $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent random variables. $\mathbf{X}_1$ is uniformly distributed over $(-1, 1)$ and $\mathbf{X}_2$ is exponentially distributed with parameter $\lambda = 1$.

   (a) Find the pdfs of the RVs $\mathbf{W} = \mathbf{X}_1 + \mathbf{X}_2$ and $\mathbf{Z} = \mathbf{X}_1/\mathbf{X}_2$.

   (b) Find the joint pdf of $\mathbf{W}$ and $\mathbf{Z}$.

5. Let $(\mathbf{X}, \mathbf{Y})$ be uniformly distributed on the interior of the square with vertices at $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$.

   (a) Are $\mathbf{X}$ and $\mathbf{Y}$ independent? Are they uncorrelated?

Now let $(\mathbf{X}, \mathbf{Y})$ be uniformly distributed on the interior of another square with vertices at $(2,0), (1,1), (0,0)$ and $(1,-1)$.

(b) Determine whether $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated or independent in this case.

(c) What valuable lesson do the above two examples serve to reinforce with regard to the relation between independence and correlation?

(d) Are $\mathbf{X}+\mathbf{Y}$ and $\mathbf{X}-\mathbf{Y}$ uncorrelated or independent in both of the above cases?

6. If $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are independent random variables having identical density functions $f(u) = e^{-u}$, $0 < u < \infty$, derive the joint distribution of $\mathbf{U} = \mathbf{X} + \mathbf{Y}$, $\mathbf{V} = \mathbf{X} + \mathbf{Z}$ and $\mathbf{W} = \mathbf{Y} + \mathbf{Z}$.

7. Let $\mathbf{X}$ and $\mathbf{Y}$ be Gaussian random variables with $E[\mathbf{X}] = 1, E[\mathbf{Y}] = -1, \mathrm{Var}[\mathbf{X}] = 2, \mathrm{Var}[\mathbf{Y}] = 1$ and $\rho_{X,Y} = 0.5$.

(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

(b) If $\mathbf{U} = 2\mathbf{X} - 3\mathbf{Y}$ and $\mathbf{V} = 2\mathbf{X} + 3\mathbf{Y}$, what is $\mathrm{cov}(\mathbf{U}, \mathbf{W})$?

(c) What is the joint pdf of $\mathbf{U}$ and $\mathbf{W}$?