University of Illinois

Problem Set 12

ECE 313
Spring 1998

Assigned : Friday, April 20, 1998
Due : Monday, April 27, 1998
Reading : Ross, Chapter 6

1. The joint pmf of $\mathbf{X}$ and $\mathbf{Y}$ is

$$p_{X,Y}(i,j) = \begin{cases} 2^{-(i-1)}\,3^{-j}, & i = 1,2,\ldots;\, j = 1,2,\ldots \\ 0, \quad \text{otherwise} \end{cases}$$

   (a) Find the marginals of $\mathbf{X}$ and $\mathbf{Y}$. What kinds of random variables are these?

   (b) Find the conditional pmf of $\mathbf{X}$ given that $\mathbf{Y} = k$.

   (c) Are $\mathbf{X}$ and $\mathbf{Y}$ independent? Can you arrive at this conclusion from your answer to part **(b)**?

2. Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be three independent Bernoullis RVs with parameter p, i.e., $\mathbf{X}_i$ takes on two values, 1 and 0 with probabilities p and $1-p$, respectively, for all i.

   (a) What kind of random variable is $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3$?

   (b) Can you generalize to the sum of n independent, identically distributed Bernoulli random variables, each with parameter p?

   (c) Using this interpretation, if $\mathbf{X}$ and $\mathbf{Y}$ are two independent binomial RVs with parameters (m,p) and (n,p), what is the pmf of their sum?

3. Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly continuous independent exponential random variables, each with parameter λ.

   (a) Find the pdf of the RV $\mathbf{A} = \min\{\mathbf{X}, \mathbf{Y}\}$.

   (b) Find the pdf of the RV $\mathbf{B} = \max\{\mathbf{X}, \mathbf{Y}\}$.

   (c) Evaluate the pdf of the RV $\mathbf{Z} = \mathbf{A} + \mathbf{B}$. How is this similar to the pdf of $\mathbf{X} + \mathbf{Y}$? Why? (You should get a very interesting answer.)

4. Calvin knows that Susie walks past "Dead Man's Perch" at around noon on Sunday, but he isn't sure exactly when. In reality, the time at which Susie crosses "Dead Man's Perch" is uniformly distributed between 11:45 and 12:15. In his eagerness to pelt Susie with his best slushball ever, Calvin decides to get to "Dead Man's Perch" early and ambush Susie. Due to various feline factors, his time of arrival is uniformly distributed between 11:55 and 12:15, and is of course independent of Susie's.

   (a) What is the probability that Calvin arrives at "Dead Man's Perch" before Susie does?

   (b) If he decides to wait no more than 10 minutes for her, what is the probability that he gets to fling his slushball at her?

5. Random variables $\mathbf{X}$ and $\mathbf{Y}$ have a uniform joint density on the square bounded by the following four corners: $(1,0), (0,1), (-1,0)$ and $(0,-1)$.

   (a) Calculate the marginal pdfs of $\mathbf{X}$ and $\mathbf{Y}$. Are $\mathbf{X}$ and $\mathbf{Y}$ independent?

   (b) Compute $E[X]$ and $\mathrm{Var}[X]$.

   (c) Calculate the pdfs of the random variables $\mathbf{A} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{B} = \mathbf{XY}$.

   (d) Calculate the pdf of the random variable $\mathbf{C} = \mathbf{X}/\mathbf{Y}$.

6. If the random variables $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are independent, and chosen uniformly in the interval $[-1, 1]$,

   (a) Find the joint CDF of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

   (b) What is the probability that all of the roots of the equation $\mathbf{A}x^2 + \mathbf{B}x + \mathbf{C} = 0$ are real?

7. **EXTRA CREDIT, worth 20pts:**
   Three points are chosen at random in the interval $[0, 1]$.

   (a) Find the pdf of the middle point.

   (b) Find the pdf of the distance of the third point from 1.