University of Illinois Spring 1998

ECE 313: Problem Set #11

Assigned: April 10, 1998
Due: April 20, 1998
Reading: Ross, sections 1,2,3 of Chapter 6

Noncredit Exercises: (Do not turn these in) All the exercises are from the Ross textbook: pp. 291–294, 1–5, 10, 11–13, 21; pp. 296–297, 1–10, 13–19.

1. Suppose that two cards are drawn at random from a deck of 52 cards. Let X be the number of aces obtained and let Y be the number of queens obtained.

(**a**) Find the joint probability mass function of X and Y.

(**b**) Find the marginal probability mass function of X and that of Y.

For those who never play card games: there are 4 aces and 4 queens in a deck of 52 cards.

2. The discrete random variables X and Y have joint p.m.f. $p_{X,Y}(u,v)$ given by

$v \downarrow / u \rightarrow$	0	1	3	5
-1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	0
3	$\frac{1}{6}$	$\frac{1}{12}$	0	$\frac{1}{12}$
4	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

(**a**) Find the joint CDF of X and Y. Specify the value of $F_{X,Y}(u,v)$ for all u and v.

(**b**) Find the marginal probability mass functions $p_X(u)$ and $p_Y(v)$ of X and Y.

(**c**) Find $P\{X \le Y\}$ and $P\{X+Y \le 8\}$

3. Is the following function

$$F(u,v) = \begin{cases} 0 & u+v<1 \\ 1 & u+v \ge 1 \end{cases}$$

a valid joint CDF. Why or why not? Prove your answer and show your work.

4. The random variables X and Y have joint probability density function

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-(u+v)} & 0<u<v<\infty \\ 0 & \text{otherwise} \end{cases}$$

(**a**) Sketch the uv plane with an indication of the region where $f_{X,Y}(u,v)$ is nonzero.

(**b**) Find the joint CDF of X and Y. Specify the value of $F_{X,Y}(u,v)$ for all u and v.

(**c**) Find the marginal CDFs of X and Y by setting, respectively, v and u to $+\infty$ in the answer to part (**b**).

(**d**) Find the marginal probability density functions of X and Y by:

   i. integrating the joint probability density function
   ii. differentiating the marginal CDFs found in part (**c**)

   Compare the two answers obtained in (i) and (ii).

(**e**) Find $P\{Y > 3X\}$.

(**f**) For $\alpha > 0$, find $P\{X + Y \leq \alpha\}$.

(**g**) Use the result in part (**f**) to determine the p.d.f. of the random variable $Z = X + Y$.

5. The random variables X and Y are uniformly distributed on the unit circle $u^2 + v^2 \leq 1$.

(**a**) Write down the joint probability density function of X and Y.

(**b**) Find the marginal probability densities of X and Y.

The following two problems are optional for extra credit. Try them: each is worth extra 10 points in the sum of your homework scores.

6. The joint probability density function of the variables X, Y, Z is given by

$$f_{X,Y,Z}(u,v,w) \;=\; \begin{cases} \dfrac{c\cdot(u+v+w)^2}{uvw} & 0 < u,v,w < 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find $P\{X < Y < Z\}$.

7. Let X and Y be discrete random variables taking on the real values $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$, respectively, and let $p_{i,j} = P(\{X = u_i\} \cap \{Y = v_j\})$. Show that X and Y are independent if and only if the rank of the joint p.m.f. matrix $P = [p_{i,j}]$ is equal to 1.

Linear algebra reminder: rank of a matrix = number of linearly independent rows = number of linearly independent columns.