University
of Illinois

Problem Set 10

ECE 313
Spring 1998

Assigned : Friday, April 3, 1998
Due : Monday, April 13, 1998
Reading : Ross, Chapter 4,5

1. **[15pts]** A number is chosen uniformly in $(0, 1)$.

   (a) Find the pdf/pmf of the negative of its logarithm (to base e).

   (b) Find the pdf/pmf of the second decimal of its square root. (**Hint:** If the first decimal digit of a number $z \in (0, 1)$ is m and its second digit is k, then $10m + k \leq 100z < 10m + k + 1$.)

2. If the continuous RV $\mathbf{X}$ has CDF $F_X(u)$ and pdf $f_X(u)$, what are the CDF and pdf/pmf of the RVs (i) $\alpha|\mathbf{X}| + \beta$ (ii) $\alpha\,\mathrm{sign}(\mathbf{X})$ $(\mathrm{sign}(\mathbf{X}) = +1, \mathbf{X} \geq 0;\ = -1, \mathbf{X} < 0)$, where α, β are real numbers and $\alpha \neq 0$? Now assume that $\mathbf{X} \sim N(0, a)$. What are the CDFs and pdfs/pmfs from parts (i) and (ii) above? Sketch or plot (using a computer) these functions.

3. In class we noted that the mean and variance (or some other moments) of some random variables might either blow up to infinity or not exist at all! Here is one such example: A standard Cauchy RV has density function

$$f_X(u) = \frac{1}{\pi(1 + u^2)}, \qquad -\infty < u < \infty$$

   (a) Prove that $E[\mathbf{X}]$ doesn't exist and $Var[\mathbf{X}]$ is infinite.

   (b) Show that $1/\mathbf{X}$ is a standard Cauchy RV.

4. Let $\mathbf{X}$ be a continuous RV with CDF $F_X(u)$.

   (a) Define another random variable $\mathbf{Y}$ by $\mathbf{Y} = F_X(\mathbf{X})$. Show that $\mathbf{Y}$ is uniformly distributed over the interval $[0, 1]$, **regardless** of what $F_X(u)$ is.

   (b) Now let's do the reverse: We wish to generate a continuous RV $\mathbf{X}$ with a specified distribution (CDF) $\mathcal{F}(u)$, and all we are given is a uniform RV on $[0, 1]$. Define the random variable $\mathbf{X}$ by $\mathbf{X} = \mathcal{F}^{-1}(\mathbf{Y})$ (remember that $\mathcal{F}$ is monotonic increasing on $[0, 1]$), where $\mathbf{Y}$ is uniform on $[0, 1]$. Show that $\mathbf{X}$ has CDF $\mathcal{F}(u)$. (This is a widely used method to generate random variables of desired distributions, for example in computer simulations. Try it out!)

5. **[20pts]** A generic communication system involves the transmission of a *known* signal s with amplitude α over a channel that corrupts the signal with additive noise, which is typically modeled as a Gaussian RV $\mathbf{X} \sim N(0, \sigma^2)$. Thus, the received signal is a random variable given by $\mathbf{R} = \alpha s + \mathbf{X}$.

   (a) Find the CDF and pdf of $\mathbf{R}$.

   (b) Find the CDF and pdf of the RV $\mathbf{Y} = e^{\mathbf{R}}$, and sketch/plot them ($\mathbf{Y}$ is called a *lognormal* RV since $\ln \mathbf{Y}$ is a normal (or Gaussian) RV. This random variable frequently raises its ugly head in the realm of wireless communications).

(c) In all modern-day electronics, the received continuous waveform is digitized by passing it through an analog-to-digital (A/D) converter that quantizes the waveform. In this part of the problem we will design two different quantizers for our received signal $\mathbf{R}$. The input/output relations of the two quantizers are given in the diagram below. The quantizer on the left, Quantizer A, has 3 levels, while the one on the right, Quantizer B, has 4 levels. The transition points in both cases occur symmetrically about $E[\mathbf{R}]$ (the mean of the received signal) i.e., at $E[\mathbf{R}] - c$ and $E[\mathbf{R}] + c$.

   i. What kind of RV is $\mathbf{Y}$, the output of the quantizer? Find and sketch its CDF and pdf/pmf for both Quantizer A and Quantizer B.

   ii. Now let us choose a value for c, thereby truly *designing* our quantizers. One way to choose c is to minimize the **mean squared error** between the input to the quantizer $\mathbf{R}$ and output from the quantizer $\mathbf{Y}$. That is, c is chosen so as to minimize $E[(\mathbf{R} - \mathbf{Y})^2]$, where the expectation is with respect to $f_R(r)$. Find this value of c for Quantizers A and B.

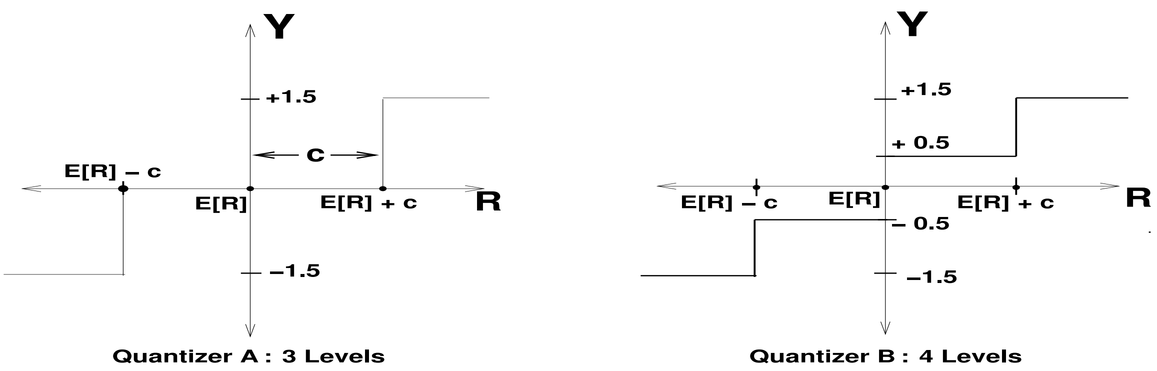

Figure 1: Three and four level Quantizers: Quantizer A is a 3-level quantizer; Quantizer B is a 4-level quantizer. The transition points in both cases are symmetrically spaced about $E[\mathbf{R}]$.

6. (a) It's Easter time and the jellybean jar in Calvin's house contains beans of n different colors in **equal numbers**. To keep him occupied, Calvin's mom tells him that he can eat a certain number of beans from the jar according to the following strategy: he should pick beans from the jar one by one, record their color and put them back in the jar. The moment he gets a bean whose color he has picked before, he stops and eats one bean for each color that he picked, **including the one repetition**. Let $\mathbf{X}$ be the RV denoting the number of beans that Calvin picks. What is the **maximum** value that $\mathbf{X}$ can take? Find the pmf of $\mathbf{X}$.

   (b) Calvin soon gets bored of this game and so his dad gives him this slightly more "interesting" game to play. Calvin is again supposed to start picking beans from the jar, recording their color and putting them back in the jar. This time, he stops only when he has seen **at least one bean of each color**. We let $\mathbf{Y}$ be the RV denoting the number of beans that Calvin picks this time. What is the **minimum** value that $\mathbf{Y}$ can take? Find the pmf of $\mathbf{Y}$.

   (c) **Extra credit worth 25pts each:** In both the above games find the expected number of beans that Calvin will get to eat. (**Hint:** It might be easier in this case to *not* try to calculate this from first principles using the pmf of $\mathbf{X}$ or $\mathbf{Y}$.)