University
of Illinois

Problem Set 6

ECE 313
Spring 1998

Assigned : Friday, February 27, 1998
Due : Monday, March 9, 1998
Reading : Ross, Chapter 3
About the problems : These problems are based on independent trials and hypothesis testing.

1. Prof. Max L. Hood wishes to determine whether a student is excellent, good or average, which he defines as hypotheses H_0, H_1, and H_2 respectively. Let E, G, A, and B denote the events that a student's performance on an exam is excellent, good, average, or bad. From statistics collected over the years, Prof. Hood knows that $P_0(E) = 0.7, P_0(G) = 0.2, P_0(A) = 0.08$, and $P_0(B) = 0.02$, that is, an excellent student has a 70% chance of faring excellently, a 20% chance of having a good performance, an 8% chance of having an average performance, and a 2% chance of performing poorly on the exam. Similarly,

$$P_1(E) = 0.15, P_1(G) = 0.7, P_1(A) = 0.1, P_1(B) = 0.05$$
$$P_2(E) = 0.1, P_2(G) = 0.2, P_2(A) = 0.6, P_2(B) = 0.1$$

   (a) For each of the four possible results of a student's exam, determine Prof. Hood's maximum likelihood decision as to whether the student is excellent, good, or average.

   (b) What is the probability that an excellent student is mistakenly labeled as average by the professor? What is the probability that an average student is labeled as good? How about the probability that a poor student is labeled above average (i.e., good or excellent)?

   (c) If it is known that 15% of the students are excellent, 50% are good, and the remaining 35% are average (i.e., $P(H_0) = 0.15, P(H_1) = 0.5, P(H_2) = 0.35$), what is Prof. Hood's probability of mis-classifying the students?

   (d) Prof. Mini P. Err decides to use a Bayes' decision rule to classify the students. What is the decision rule she uses? What is the probability that she mis-classifies the students?

   (e) In another class, 85% of the students are excellent and 15% are good (no average students). What is Prof. Err's decision rule in this case based on the four possible results of the student's exam?

   (f) For what range of values of $P(H_0)$ does Prof. Err's rule always decide that H_0 is true? Note that the decision rule should be applicable regardless of the values of $P(H_1)$ and $P(H_0)$, provided they lie strictly between 0 and 1.

2. Baskets A, B, and C contain two black balls each, and in addition, contain respectively two, three, and five white balls. A basket is chosen at random and a ball drawn from it.

   (a) What is the maximum likelihood (ML) rule for deciding which basket has been chosen based on observation of the color of the ball drawn?

   (b) Suppose the ML rule is being used, and the baskets are chosen with probabilities $P(A) = 1/2, P(B) = 1/6, P(C) = 1/3$. Find $P(E)$, the probability of making a wrong decision.

(c) Find the Bayes' rule for the case of the prior probabilities in part (b). Show that its error probability is lower than that of the ML decision rule.

3. The probability that you can hear the sound of a pin dropping onto a table during a long-distance telephone call is p_0 if the call is made through the AT&T network, and p_1 if it is made over Sprint's network. Assume that both p_0 and p_1 are quite small (who are they trying to fool anyway?) and that $p_1 > p_0$. You call Candice Bergen from a payphone controlled by either AT&T or Sprint (but you don't know which), and she agrees to start dropping pins one by one until you hear the sound of one dropping. Suppose the first time you actually hear the sound of a pin hitting the table is at pin number 30.

   (a) Let H_0 and H_1 denote the hypotheses that the call is carried by AT&T and Sprint respectively. What is the likelihood ratio (LR) for the observation that the 30^{th} pin dropped is the first one heard?

   (b) The ML decision compares LR to the threshold 1 and announces in favor of H_0 or H_1 according as $LR < 1$ or $LR > 1$. Show that this decision rule can be expressed in terms of a threshold test on k, the number of the first pin that you heard being dropped.

   (c) If $p_0 = 0.02$ and $p_1 = 0.05$, what is the ML decision if the 30^{th} pin dropped is the first one heard? Find the false alarm probability for this case (i.e., the probability of mistaking the line for an AT&T one, when it is actually a Sprint one). Also find the probability of missed detection (i.e., the probability of mistaking the line for Sprint, when it is actually owned by AT&T).

   (d) If AT&T, the big monster company, owns 90% of all long-distance lines, and Sprint owns the remaining 10%, it is reasonable to assume that $P(H_0) = 0.9$ and $P(H_1) = 0.1$. Now what is the Bayesian decision rule if the 30^{th} pin is the first to be heard?

   (e) Find the average probability of error for both decision rules, ML and Bayes.

4. **Modified Binary Erasure Channel:** Consider the noisy discrete communication channel illustrated by the diagram below. Let H_0 and H_1 represent the hypotheses that a 0 or 1 was transmitted respectively.

(a) If $P[0] = 0.7$ and $P[1] = 0.3$, determine the ML and the optimum (from the point of view of minimizing the probability of error) decision rules (of assigning a, b, c to $0, 1$) and the resulting probabilities of error for both rules.

(b) How many possible decision rules are there? Plot the probability of error for all the decision rules versus $P[0]$ on a graph. Clearly mark the ML decision rule.

(c) Each decision rule has a maximum probability of error, which occurs for some least favorable *a priori* probability $P[0]$. The decision rule which has the smallest maximum probability of error is called the minimax decision rule. Which of the eight rules is minimax?

5. Calvin and Hobbes decide to play a game of coin-tossing, with a fair coin being tossed once - Calvin wins if it turns up Heads and Hobbes wins if it turns up Tails. Thus, each should have a probability of 1/2 of winning. However, Hobbes knows that Calvin will definitely cheat and use a biased coin with $P(\text{Heads}) = p \neq 1/2$. It is obviously in his best interest to propose a strategy that will *simulate* a fair coin toss at every toss. This is what he proposes:

   (a) Calvin and Hobbes each toss the coin once. Calvin wins if he tosses a Head *and* Hobbes tosses a Tail. Hobbes wins if he tosses a Head *and* Calvin tosses a Tail. What are the probabilities of these events? What is the probability that neither wins?

   (b) If Calvin's and Hobbes' tosses match (i.e., both Heads or both Tails) each tosses again. The winning requirement remains the same. This goes on and on until ultimately either Calvin or Hobbes wins. Show that $P(\text{Calvin wins}) = P(\text{Hobbes wins}) = 1/2$. (Rats, Calvin is foiled again! This problem shows that you can simulate a fair coin toss with a biased coin! And the result doesn't depend on p, provided $0 < p < 1$. Amazing, huh?)

6. This problem is a continuation of the above problem, i.e., repeated coin tosses with a biased coin with $P(\text{Heads}) = p \neq 1/2$. Different parts ask you to assume that the value of p is known or unknown.

   (a) Suppose that Calvin and Hobbes agree to toss the coin 100 times. Calvin is certain that $p = 0.75$. If the game is to bet on the number of heads that will occur in these 100 tosses, what should Calvin bet on? What is the probability that he is correct, i.e., what is the probability that the 100 coin tosses *do* result in the number of Heads he "predicted"?

   (b) Now Hobbes does not know the value of p, but wants to estimate it based on the 100 tosses. He sees that Heads occur on 80 of the tosses and Tails on 20. What should he estimate p to be? Why?

   (c) Again, assuming that the value of p is unknown to Hobbes, if the first Tail occurred on the 10^{th} toss what is the ML estimate of the value of p? That is, what value of p maximizes the probability of a Tail occurring for the first time on the 10^{th} toss? How does his estimate of p change if he sees that the second Tail occured on the 16^{th} toss?

   (d) If Hobbes is asked to bet on the number of tosses required to observe a Tail for the first time, what number should he bet on? Why? Given that $p = 0.75$ as before, what is the probability that the number he chose is actually the number of tosses required for a Tail to be seen?

(e) Again, given that $p = 0.75$, what should Hobbes bet as the number of tosses required to see a Tail 20 times? What is the probability that he is correct, that is, what is the probability that a Tail occurs for the 20^{th} time on the number of tosses he predicts?

(f) If the 20^{th} Tail was observed on the 95^{th} toss, what is the ML estimate of p? Compare this part with part (b), where 20 Tails occur in the entire 100 tosses. This part is a special case of part (b) because one way that Hobbes could have observed 20 Tails in 100 tosses is for the 95^{th} toss to be the 20^{th} Tail and the remaining 5 tosses to be Heads. Why is the ML estimate of p in part (f) different from that in part (b)?