University of Illinois

Problem Set 3

ECE 313
Spring 1998

Assigned : Friday, February 6, 1998
Due : Monday, February 16, 1998
Reading : Ross, Chapter 2 and Chapter 3.1–3.2
About the problems : These problems are based on axiomatic probability, equally likely outcomes, and elementary notions of conditional probability.

1. Let A, B, C denote three events defined on the same sample space. Prove that

   (a) $(P(A) + P(B) + P(C))/3 \leq P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.

   (b) Bonferroni's inequality: $P(ABC) \geq P(A) + P(B) + P(C) - 2$.

   (c) Generalized Bonferroni's inequality: Given n events $A_1, A_2, \ldots, A_n$, use induction to show that

   $$P(A_1 A_2 \ldots A_n) \geq P(A_1) + P(A_2) + \cdots P(A_n) - (n-1).$$

2. Find $P\left(A \cup (B^c \cup C^c)^c\right)$ in each of the following cases:

   (a) A, B, C are mutually exclusive events and $P(A) = 3/7$.

   (b) $P(A) = 1/2, P(BC) = 1/3, P(AC) = 0$.

   (c) $P\left(A^c \cap (B^c \cup C^c)\right) = 0.65$.

3. **Matching Problem :** A router receives 8 packets that need to be routed to stations A, B, C, D, E, F, G and H, each station receiving exactly one packet. Due to a lightning strike at the router site, it cannot parse the header information in a packet and hence, with equal probability, sends a given packet to any one of the 8 stations.

   (a) Find the probability that the router gets at least 6 packets routed correctly?

   (b) Given that Station A does *not* receive the correct packet, what is the probability that *none* of the stations receive their designated packets?

4. (a) An ordinary deck of 52 cards is shuffled and then cards are overturned one at a time till the first ace appears. Given that the first ace is the 3^{rd} card to appear, what is the conditional probability that the card following it is (i) Ace of Spades (ii) Two of Clubs? How do the conditional probabilities change if the first ace is the 20^{th} card?

   (b) This is a modification of the above part. A deck of cards is shuffled and then the cards are overturned one by one till the first ace is encountered. What is the probability that the next card is an Ace of Spades? What is the probability that the next card is a Two of Clubs? [This part is more subtle than the previous one.]

5. (a) Consider 3 baskets. Basket A contains 2 white and 4 red balls; basket B contains 8 white and 4 red balls; and basket C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from basket A was white, given that exactly 2 white balls were selcted?

   (b) Same problem statement as above: If a genie blindfolds you and then hands you a ball picked at random from one of the baskets, what is the probability that it came from basket A, given that it is white?

6. **Ternary Channel:** A communication system transmits three signals, s_1, s_2, and s_3, with equal probabilities. The reception is corrupted by noise, causing the transmission to be changed according to the following table of conditional probabilities:

		Receive, j		
		s_1	s_2	s_3
Send, i	s_1	0.8	0.1	0.1
	s_2	0.05	0.9	0.05
	s_3	0.02	0.08	0.9

   For example, if s_1 is sent, the probability of receiving s_3 is 0.1. The entries list the probability of s_j received, given that s_i is sent. Compute the probabilities that s_1, s_2, s_3 are received. Compute the probabilities $P[s_i \text{ sent} \mid s_j \text{ received}]$ for $i, j = 1, 2, 3$.

7. **Dilbert's Ruin?** Dilbert-the-Geek is a contestant on Monty Hall's "Let's Make a Deal" and in the final round, Monty shows him 3 curtains. One of the curtains conceals a prize, the other two, junk. Dilbert selects a curtain at random. Monty, who knows where the prize is, opens one of the remaining curtains to reveal junk. He now offers Dilbert the "new improved deal" — Dilbert can either stick to his original guess or he can switch. Amid roars of "Switch, you idiot!" from Dogbert-the-Conqueror, and "Stay, you moron!" from Wally-the-Loser, Dilbert ponders about his choices and the meaning of life.

   (a) We saw in class that it was advantageous for Dilbert to actually switch at this point. We can safely assume that Dilbert figures this out too.

   In a follow-up to this game, Monty calls Wally down from the audience to join Dilbert and asks each of them to pick a curtain. Monty opens one of the curtains *that was picked* revealing junk behind it and sends that person back to the audience (this person is most probably Wally-the-Loser). The other player is now offered the choice of staying with his original choice or switching to the remaining curtain.

   (b) Compute the probability of his winning under both strategies — staying put OR switching. Is this the same as before?

   (c) What is the probability that Dilbert wins the prize? What is the probability that Wally wins the prize?

Note: Monty always opens one of the two curtains picked by his contestants. In some cases, one of the two curtains conceals the prize and so he is forced to open the other. If neither curtain conceals the prize then he opens one at random.

(d) **For extra credit, equal to a complete problem:** Having seen that Dilbert is possessed of extraordinary analytical capabilities, Monty decides to put him to the ultimate test — the *4 curtain test*, on an adjoining stage. There are now *four* curtains, one concealing a prize and the other 3 concealing junk. Dilbert picks a curtain and Monty opens one of the curtains revealing junk (again, remember that Monty knows where the prize is). Dilbert again has the choice of sticking to his decision or switching. Whether he stays or switches, Monty opens *another* curtain once Dilbert has made his choice (so now he has opened 2 curtains, one of which could have been Dilbert's first choice, if he switched) and yet again offers him the "new improved deal". Should Dilbert switch or stay put now (remember, at this stage there are two curtains that Monty has opened and Dilbert has two curtains left to choose between)? How does the problem generalize to n curtains and one prize?