University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 313
Hour Exam # 2

Wednesday, April 15, 1998
7:00 p.m. — 8:00 p.m.
Room 112 Chemistry Annex Building

Name ______________________________

University ID Number ______________________________

Signature ______________________________

Section:

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INSTRUCTIONS

This exam is closed book and closed notes, except that two 8.5″ × 11″ sheet of notes (both sides) are allowed. Calculators, laptop computers, tables of integrals, etc. may not be used.

The exam consists of 4 problems worth a total of 100 points. Write your answers in the spaces provided. Show *all* your work, except in the true/false problem. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem.

Grading

1. 18 points	________
2. 28 points	________
3. 34 points	________
4. 20 points	________
TOTAL	________

Problem 1 (18 points)

In the following statements, X and Y are generic continuous random variables with well-defined finite mean and variance. The functions $F_X(u)$ and $f_X(u)$ denote the cumulative distribution function and the probability density function of X, respectively.

True	False	
☐	☐	$f_X(u) \leq 1$ for all u
☐	☐	$P\{a < X < b\} = P\{e^a < e^X < e^b\}$ for all real a and b
☐	☐	$P\{X^2 - 2X + 1 < -1\} = 0$
☐	☐	If $\mathcal{Z} \sim$ Poisson with parameter λ, then $E[(\mathcal{Z}+1)^2] = (\lambda+1)^2$
☐	☐	If X and Y have the same variance, then $E[X^2] = E[Y^2]$
☐	☐	If $Y = \lvert X \rvert$, then $E[Y] \geq E[X]$

Check the appropriate box for each of the eight statements above. Answer True if the statement is true for **all** such X and Y. Otherwise, answer False. **No justification is required**. However, so as not to give points for random guessing, you will receive $+3$ points for correct answer, 0 points for no answer, and -3 points for wrong answer.

Problem 2 (28 points)

The received signal $\mathcal{R}$ in a communication system corresponds to one of two hypotheses:

$$\begin{aligned} H_0 : \quad \mathcal{R} &= \mathcal{S}_0 \\ H_1 : \quad \mathcal{R} &= \mathcal{S}_1 \end{aligned}$$

The random variables $\mathcal{S}_0$ and $\mathcal{S}_1$ are both Gaussian, with the same variance $\sigma^2 = 4$. It is known that $E[\mathcal{S}_0] = 0$ while $E[\mathcal{S}_1] = 1$. In other words, we have $\mathcal{S}_0 \sim \mathcal{N}(0, 2^2)$ and $\mathcal{S}_1 \sim \mathcal{N}(1, 2^2)$.

The received signal $\mathcal{R}$ is now passed through a quantizer to produce one of the following three possible observations:

$$\begin{aligned} \mathcal{A}_1 : &\quad \mathcal{R} < -1 \\ \mathcal{A}_2 : &\quad -1 \le \mathcal{R} < +1 \\ \mathcal{A}_3 : &\quad \mathcal{R} \ge +1 \end{aligned}$$

Some of the values of the cumulative distribution function $\Phi(\cdot)$ of the standard Gaussian random variable $\mathcal{N}(0,1)$ are given below:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\Phi(x)$	0.500	0.540	0.579	0.618	0.655	0.692	0.728	0.758	0.788	0.816

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\Phi(x)$	0.841	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971

a. Calculate the probabilities of the events $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ under the two hypotheses, and complete the following likelihood matrix.

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
H_0			
H_1			

b. Find the maximum-likelihood decision rule for the likelihood matrix computed in part (**a**), and indicate it on the matrix below.

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
H_0			
H_1			

c. Under this rule, what is the probability of deciding H_0 given that H_1 is true?

$P(1 \rightarrow 0) \;=$

d. Now suppose that the prior probabilities $P(H_0) = 0.2$ and $P(H_1) = 0.8$ are known. Find the Bayes decision rule for this case, and indicate it on the matrix below.

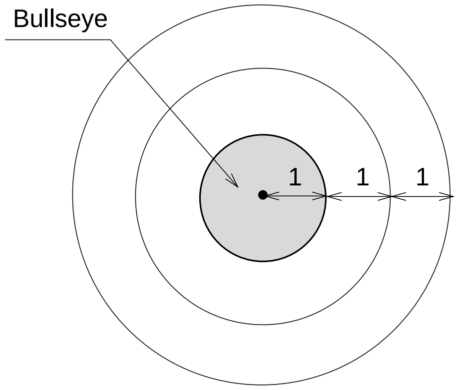
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$
H_0			
H_1			

e. What is the average probability of error for the above Bayes decision rule?

$P_e \;=$

Problem 3 (34 points)

Alice is a professional dart player, who is invited to play in a tournament. The tournament involves throwing darts at a dartboard, shown in the diagram below. Each contestant throws darts at the dartboard, one dart after another, as long as he or she hits the bullseye, and leaves the tournament after the first miss of the bullseye.



The bullseye is the innermost circle of radius 1 about the center of the dartboard.

Let X denote the random variable corresponding to the distance from the center that Alice hits on each dart throw. The probability density function of X is given by

$$f_X(u) = \begin{cases} \frac{4}{9}u & \text{if } 0 \le u < 1.5 \\ \frac{4}{3} - \frac{4}{9}u & \text{if } 1.5 \le u < 3 \\ 0 & \text{otherwise} \end{cases}$$

a. Let $\mathcal{A}$ be the event that Alice hits the bullseye on a given throw.

What is the probability of $\mathcal{A}$?

$P(\mathcal{A}) =$

b. Let Y denote the total number of darts Alice gets to throw in the tournament, before she leaves after her first miss. Assume that successive throws are independent trials.

Find the expected value and the variance of Y.

$E[Y] =$	$\mathsf{Var}[Y] =$

c. Alice is paid \$20 for taking part in the tournament and \$70 for each dart she gets in the bullseye. Let $\mathcal{Z}$ denote the total amount of money that Alice collects.

Find the expected value and the variance of $\mathcal{Z}$.

$E[\mathcal{Z}] =$	$\mathsf{Var}[\mathcal{Z}] =$

Problem 4 (20 points)

A continuous random variable X is uniformly distributed on the interval in $[-1, +1]$.

a. Find the probability density function of the random variable $Y = X^2$.

$$f_Y(v) = \begin{cases} & \\ & \end{cases}$$

b. Let the random variable $\mathcal{S}$ be the sign of X, that is $\mathcal{S} = 1$ if $X \geq 0$ and $\mathcal{S} = -1$ if $X < 0$. Find the mean and the variance of the random variable $\mathcal{Z} = 1 - \mathcal{S}^2$.

$E[\mathcal{Z}] =$

$\mathsf{Var}[\mathcal{Z}] =$