1. [5+5+5 points] Consider a game with probability of winning \( p = 1/3 \). If we win, we receive \$10, otherwise we pay \$2. Assume that we play the game until we win for the first time.

   (a) Find the probability that we earn \$4.

   **Solution:** To earn \$4, we need to lose 3 times (we lose \$2 every time) and then to win one time (we win \$10). The probability of this event is \((2/3)^3 \times (2/3) \times (1/3) = 8/81\).

   (b) Find the mean value of our payoff (i.e., our expected earnings).

   **Solution:** Let \( L \) denote the number of games played until we win the game for the first time. Then, \( L \sim \text{Geo}(1/3) \). Our payoff will be \(-2(L - 1) + 10\). Taking the expectation and using the fact that \( E[L] = 1/p = 3 \), our expected payoff is \(-2(3 - 1) + 10 = 6\).

   (c) Now assume that instead of stopping at the first win, we keep playing the game an infinite number of times. Let \( L_1 \) be the number of games needed until the first win. Let \( L_2 \) denote the number of games, after the first \( L_1 \) trials, until the second win. Find \( P(L_2 = 3 | L_1 = 5) \).

   **Solution:** \( L_1 \) is clearly independent of \( L_2 \) (in a Bernoulli process, all geometric random variables are independent). Therefore, \( P(L_2 = 3 | L_1 = 5) = P(L_2 = 3) = (2/3)^2 \times (1/3) = 4/27\).

2. [6+6+6+3 points] Consider the experiment of rolling two fair dice, each with 6 faces numbered 1, 2, 3, 4, 5, 6. Let \( S \) and \( P \) denote the sum and product of the numbers showing on the two dice, respectively.

   (a) Find the mean of \( P \).

   **Solution:** Let \( X_1 \) and \( X_2 \) denote the numbers showing on the two dice. We have

   \[ P\{X_i = k\} = \frac{1}{6}, \quad \text{for } i = 1, 2; \ k = 1, 2, \ldots, 6. \]

   Furthermore, \( X_1 \) and \( X_2 \) are independent. The mean of \( P \) can be calculated as

   \[ E[P] = E[X_1X_2] = E[X_1]E[X_2], \]

   while for \( i = 1, 2 \) we have

   \[ E[X_i] = \sum_{k=1}^{6} \frac{1}{6}k = \frac{7}{2}. \]

   Hence,

   \[ E[P] = \left( \frac{7}{2} \right)^2 = \frac{49}{4}. \]

   (b) Find the probability that \( S \) is even.

   **Solution:** Define the following events: \( E_i = \text{"}X_i \text{ is even"} \), \( O_i = \text{"}X_i \text{ is odd"} \) for \( i = 1, 2 \). Clearly, \( P(E_i) = P(O_i) = 1/2, i = 1, 2 \). We have \( P(\text{"}S \text{ is even"}) = P(E_1E_2) + P(O_1O_2) = P(E_1)P(E_2) + P(O_1)P(O_2) \), since \( X_1 \) and \( X_2 \) are independent. Thus,

   \[ P(\text{"}S \text{ is even"}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}. \]
(c) Find the probability that $S$ is even given that $P$ is even.

**Solution:** We have

$$P(\text{"S is even" | "P is even"}) = \frac{P(\text{"S is even"}, \text{"P is even"})}{P(\text{"P is even"})} = \frac{P(E_1E_2)}{1 - P(O_1O_2)} = \frac{1}{3}.$$

(d) Are "S is even" and "P is even" mutually independent? Justify your answer.

**Solution:** Since $P(\text{"S is even"}) \neq P(\text{"S is even" | "P is even"})$, "S is even" and "P is even" are not mutually independent.

3. [6+6+6 points] When looking for the next smartphone to buy, you narrow down to two leading brands. The first brand claims that their phone lifetime is uniformly distributed over the interval 0 to 4 years, while the second brand claims that their phone lifetime in years is exponentially distributed with parameter $\lambda = 1/2$.

(a) If you use the expectation of lifetime (the larger the better), then which brand should you pick? Justify your answer.

**Solution:** Let $X$ be the lifetime of a phone from the first brand and $Y$ be the lifetime of a phone from the second brand. We have $X \sim \text{Unif}[0, 4]$ and $Y \sim \text{Exp}(\lambda)$, where $\lambda = 1/2$. Therefore, $E[X] = (0 + 4)/2 = 2$ and $E[Y] = 1/\lambda = 2$. Hence, both brands are equally good in terms of expectation of lifetime.

(b) If you will replace your phone after 2 years anyway, then a better metric would be the probability that the phone is still working after 2 years. Which brand should you pick now? Justify your answer.

**Solution:** Using the pdf and properties of uniform and exponential random variables we have

$$P(X > 2) = \int_{2}^{4} \frac{1}{4} du = \frac{1}{2},$$

$$P(Y > 2) = e^{-\lambda^2} = e^{-1} = \frac{1}{e}.$$  

Hence, $P(X > 2) > P(Y > 2)$, i.e., you should pick the first brand.

(c) If your mother only wants to replace her phone after 5 years, then which brand would you pick for her? Justify your answer.

**Solution:** Only phones from the second brand have positive probability of working after five years. Hence, you should choose the second brand for your mother.

4. [4+6+4 points] Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 4$ per hour. Let $N_t$ denote the number of buses arriving in the time interval $[0, t]$. Recall that for a fixed $t > 0$, $N_t$ is a Poisson random variable with parameter $4t$.

(a) Find the probability that no bus arrives in the first $t = 0.25$ hours. Provide your answer in terms of $e$.

**Solution:** The probability is given by $P[0.25 = 0] = e^{-\lambda t} = e^{-4\times0.25} = e^{-1}$.

(b) Find the conditional probability that there is 1 arrival in the interval $(0.5h, 1h]$ given that there are 2 arrivals in the interval $[0, 1h]$. Here, ‘$h$’ denotes ‘hours’.

**Solution:** Suppressing ‘$h$’ for notational convenience, let $A$ be the event that there is 1 arrival in the time interval $(0.5, 1]$, $B$ the event that there are 2 arrivals in the interval $[0, 1]$, and $C$ the event that there is 1 arrival in the interval $[0, 0.5]$. Then, the conditional
probability is given by

\[
\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A \cap C]}{\mathbb{P}[C]} = \frac{\mathbb{P}[A|C]}{\mathbb{P}[C]}
\]

\[
= \frac{e^{-4\times 0.5(4\times 0.5)} e^{-4\times 0.5(4\times 0.5)^{1/2}}}{1!} = \frac{e^{-4\times 1(4\times 1)^2}}{2!} = \frac{e^{-4\times 2^1} e^{-2^2}}{2!} = \frac{4e^{-2} e^{-2}}{2!} = \frac{1}{2}
\]

(c) Let \( X \) denote the number of arrivals in \([0, 1h]\) and let \( Y \) denote the number of arrivals in \((1h, 2h]\). Find \( \mathbb{P}(Y = 2|X = 1) \).

**Solution:** Since \( X \) and \( Y \) are independent and \( X \) and \( Y \) have identical distributions, \( \mathbb{P}(Y = 2|X = 1) = \mathbb{P}(Y = 2) = \mathbb{P}[N_1 = 2] = e^{-\lambda t} \frac{(\lambda t)^2}{2!} = e^{-4}(4)^2 = 8e^{-4} \).

5. **[4+4+6 points]** Consider the following \( s-t \) network, where link \( i \) fails independently with probability \( p_i \):

![Network Diagram]

Denote by \( q_i = 1 - p_i \) the probability that link \( i \) works.

(a) Let \( Y \) denote the capacity of the network, i.e., the maximum flow rate from \( s \) to \( t \). What are the possible values of \( Y \)?

**Solution:** \( Y \) takes values in the set \( \{0, 5, 10\} \).

(b) Compute \( \mathbb{P}(Y = 5) \).

**Solution:** \( Y = 5 \) if all links work except for link 4. Therefore, \( \mathbb{P}(Y = 5) = q_1 q_2 q_3 p_4 q_5 \).

(c) Compute the probability of network outage, which corresponds to the event that at least one link fails along every \( s-t \) path.

**Solution:** The network fails if either link 1 or 5 fail, which happens with probability \( p_1 + p_5 - p_1 p_5 \).

If links 1 and 5 work, then the network fails if both link 4 and the serial link 2-3 fails, which has probability \( q_1 q_5 p_4 p_{2,3} \). Here, \( p_{2,3} \) denotes the probability that the serial link 2-3 fails, which is given by \( p_{2,3} = p_2 + p_3 - p_2 p_3 \).

Therefore, we have

\[
\mathbb{P}(\text{outage}) = \mathbb{P}(Y = 0) = p_1 + p_5 - p_1 p_5 + q_1 q_5 p_4 (p_2 + p_3 - p_2 p_3).
\]

6. **[12+12 points]** The two parts of this problem are unrelated.

(a) A blind man waits at a bus stop serviced by the buses A and B. He plans to take the next bus arriving at the bus stop. Let \( X \) denote the arrival time of bus A and \( Y \) denote the arrival time of bus B. \( X \) is an exponential random variable with mean value 1 and \( Y \) is also exponential with mean value 10. Additionally, \( X \) and \( Y \) are independent. The blind man wants to take bus A. What is the probability that he takes the wrong bus?

**Solution:** He takes the wrong bus when \( Y \) is less than \( X \). \( X \) and \( Y \) are independent so the joint distribution is the product of the marginals.
\[ P(Y < X) = \int_0^\infty \int_0^\infty f_{X,Y}(u,v)dvdu \]
\[ = \int_0^\infty \int_v^\infty e^{-u}(0.1e^{-0.1v})dvdu \]
\[ = \int_v^\infty e^{-v}(0.1e^{-0.1v})dv \]
\[ = \int_0^\infty 0.1e^{-1.1v}dv \]
\[ = \frac{1}{11} \]

(b) Let \( X \) and \( Y \) be random variables with joint pdf
\[ f_{X,Y}(u,v) = \begin{cases} 8uv, & 0 \leq u \leq v, 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases} \]
Find \( f_{X|Y}(u|v) \) for any \( 0 \leq u \leq v \leq 1 \) and \( E[X|Y = v] \) for any \( 0 \leq v \leq 1 \).

**Solution:** We first note that \( f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} \).

\[ f_Y(v) = \int_0^v 8uvdu = 4vu^2|_0^v = 4v^3, \quad 0 \leq v \leq 1 \]
\[ f_{X|Y}(u|v) = \frac{8uv}{4v^3} = \frac{2u}{v}, \quad 0 \leq u \leq v \leq 1. \]

Moreover,
\[ E[X|Y = v] = \int_0^v uf_{X|Y}(u|v)du \]
\[ = \int_0^v \frac{2u^2}{v}du \]
\[ = \frac{2v^3}{3}, \quad 0 \leq v \leq 1. \]

7. [7+7+7 points] Let \( R_1 = 1 + W_1 \) denote the value of a 1Ω resistor, where \( W_1 \sim \text{Unif}[−1,1] \) is the manufacturing error. Let \( R_2 = 2 + W_2 \) denote the value of a 2Ω resistor, where \( W_2 \sim \text{Unif}[−1,1] \) is the manufacturing error as well. Assume that \( W_1 \) and \( W_2 \) are independent, i.e., \( R_1, R_2 \) are independent. Suppose that a 3Ω resistor is made by concatenating \( R_1 \) and \( R_2 \), i.e., \( R_3 = R_1 + R_2 \).

(a) Find \( E[R_3] \) and \( \text{Var}(R_3) \).

**Solution:** Since \( R_3 = R_1 + R_2 = 3 + W_1 + W_2 \), the mean is given by \( E[R_3] = E[3 + W_1 + W_2] = 3 \). The variance is given by

\[ \text{Var}(R_3) = \text{Var}(3 + W_1 + W_2) = \text{Var}(W_1) + \text{Var}(W_2) = \frac{2^2}{12} + \frac{2^2}{12} = \frac{2}{3}. \]

Here, the independence of \( W_1 \) and \( W_2 \) has been used.
(b) Assume that we bought 10 samples $X_1, X_2, \ldots, X_{10}$, of $R_1$, i.e., $X_i = 1 + W_i$ and $W_i \sim \text{Unif}[-1,1], i = 1, 2, \ldots, 10$ are independent random variables. Find the mean square error, $\mathbb{E}[(\bar{X} - \mu)^2]$, of the sample mean $\bar{X} = (X_1 + X_2 + \cdots + X_{10})/10$, where $\mu = \mathbb{E}[X_i], i = 1, 2, \ldots, 10$.

**Solution:** First, $\sigma^2 = \text{Var}(X_i) = \text{Var}(W_i) = \frac{2^2}{12} = \frac{1}{3}, i = 1, 2, \ldots, 10$. Additionally, $\bar{X}$ is unbiased, i.e., $\mathbb{E}[\bar{X}] = \mu = 1$. Then, the MSE is

$$
\mathbb{E}[(\bar{X} - \mu)^2] = \text{Var}(\bar{X}) = \frac{1}{100} \sum_{i=1}^{10} \text{Var}(X_i) = \frac{\sigma^2}{10} = \frac{1}{30}.
$$

(c) Use Markov’s inequality to upper bound $\mathbb{P}((\bar{X} - \mu)^2 \geq 0.1)$.

**Solution:** By Markov’s inequality,

$$
\mathbb{P}((\bar{X} - \mu)^2 \geq 0.1) \leq \frac{\mathbb{E}[(\bar{X} - \mu)^2]}{0.1} = 10 \times \frac{1}{30} = \frac{1}{3}.
$$

8. [10+10+10 points] Assume that if hypothesis 0 ($H_0$) is true, then the random variable $X$ takes values $-2, -1, 0, 1, 2$, each with probability $1/5$, and if hypothesis 1 ($H_1$) is true, then the random variable $X$ takes the values $-1$ with probability $1/4$, 0 with probability $1/2$ and 1 with probability $1/4$. The prior probabilities satisfy $\pi_0/\pi_1 = 2$.

(a) Find the MAP decision rule given an observation $X = k$.

**Solution:**

<table>
<thead>
<tr>
<th>$X$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$</td>
<td>$1/5$</td>
<td>$1/5$</td>
<td>$1/5$</td>
<td>$1/5$</td>
<td>$1/5$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$0$</td>
<td>$1/4$</td>
<td>$1/2$</td>
<td>$1/4$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

It is clear that for $X = 2$ and $X = -2$, $H_0$ will be selected.

For $X = -1$ and $X = 1$, we have $\Lambda(1) = \Lambda(-1) = \frac{1/4}{1/5} = \frac{5}{4} < 2$ and hence, $H_0$ will be selected as well.

For $X = 0$, we have $\Lambda(0) = \frac{1/2}{1/5} = \frac{5}{2} > 2$ and hence, $H_1$ will be selected in this case.

(b) Compute the average error probability $p_e$ of the MAP decision rule.

**Solution:** From $\pi_0/\pi_1 = 2$ we get $\pi_0 = 2/3$ and $\pi_1 = 1/3$.

$$
p_e = \pi_0 P_{\text{false alarm}} + \pi_1 P_{\text{miss}}
= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \left( \frac{1}{4} + \frac{1}{4} \right)
= \frac{27}{90}
$$

(c) Suppose that instead of an observation of $X$ we are given the sum of two independent realizations of $X$ (under the same hypothesis). If the sum of these two realizations is 0, which hypothesis will the ML decision rule declare as the true hypothesis?

**Solution:**

Denote by $X_1$ and $X_2$ the outcome of the two realizations of $X$, and by $Y$ the sum $X_1 + X_2$. 

Under $H_0$, we have
\[
P(Y = 0|H_0) = P(X_1 = 0, X_2 = 0|H_0) + P(X_1 = 1, X_2 = -1|H_0) \\
+ P(X_1 = -1, X_2 = 1|H_0) + P(X_1 = -2, X_2 = 2|H_0) \\
+ P(X_1 = 2, X_2 = -2|H_0) \\
= P(X_1 = 0|H_0)P(X_2 = 0|H_0) + P(X_1 = 1|H_0)P(X_2 = -1|H_0) \\
+ P(X_1 = -1|H_0)P(X_2 = 1|H_0) + P(X_1 = -2|H_0)P(X_2 = 2|H_0) \\
+ P(X_1 = 2|H_0)P(X_2 = -2|H_0) = 5 \left( \frac{1}{5} \right)^2 = \frac{1}{5}.
\]

Under $H_1$, we have
\[
P(Y = 0|H_1) = P(X_1 = 0, X_2 = 0|H_1) + P(X_1 = 1, X_2 = -1|H_1) + P(X_1 = -1, X_2 = 1|H_1) \\
= P(X_1 = 0|H_1)P(X_2 = 0|H_1) + P(X_1 = 1|H_1)P(X_2 = -1|H_1) \\
+ P(X_1 = -1|H_1)P(X_2 = 1|H_1) = \frac{1}{22} + 2 \frac{1}{44} = \frac{3}{8}.
\]

Since $1/5 < 3/8$, $H_1$ will be chosen as the correct hypothesis.

9. [10 points] Let $X \sim \mathcal{N}(1, 1)$. Use Chebyshev’s inequality to obtain an upper bound for $P(3 + |2X - 2|^3 \geq 67)$.

\textbf{Solution:}
\[
P(3 + |2X - 2|^3 \geq 67) = P(|2X - 2|^3 \geq 4^3) = P(|2X - 2| \geq 4) \\
\leq \frac{\text{Var}(X)}{2^2} \leq \frac{1}{4}.
\]

10. [4+8 points] Let $X \sim \mathcal{N}(0, 1)$ and $Y = aX + b$ for some real numbers $a, b$ with $a > 0$. Suppose $\sigma_Y^2 = 4$.

(a) Determine $a$.

\textbf{Solution:} Clearly,
\[
\sigma_Y^2 = a^2 \sigma_X^2 = a^2.
\]

Therefore, $a = 2$.

(b) Assume that $Y = 0$ is observed. Find the Maximum Likelihood estimate of $b$ for the value of $a$ in part (a).

\textbf{Solution:} Clearly, $Y \sim \mathcal{N}(a\mu_X + b, \sigma_Y^2) = \mathcal{N}(b, 4)$ (can be also computed using the scaling rule for pdfs). For $b$:
\[
L(b) = f_Y(0) = \frac{1}{\sqrt{8\pi}}e^{-\frac{(0-b)^2}{8}},
\]

which is maximized for $\hat{b}_{\text{ML}} = 0$.

11. [7+7+7 points] Let $X \sim \mathcal{N}(\mu, \sigma^2)$. 

(a) Define the positive random variable $Y = e^X$. $Y$ is said to have a lognormal distribution with parameters $\mu, \sigma^2$. Find $f_Y(y), y > 0$.

Solution:
\[ F_Y(y) = P(e^X \leq y) = P(X \leq \ln y) = F_X(\ln y). \]

By differentiating we obtain:
\[ f_Y(y) = f_X(\ln y)(\ln y)' = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}. \]

(b) Suppose that $Z = X + 2W$, where $W \sim \mathcal{N}(0, 1)$ is independent of $X$. Compute the unconstrained minimum MSE estimator $E[Z|X]$. Your answer should be a function of $X$.

Solution: $X, W$ are jointly Gaussian since they are independent. For the same reason, $Z, X$ are jointly Gaussian. Therefore,
\[ E[Z|X] = \mu_Z + \frac{\text{Cov}(X, Z)}{\sigma_X^2}(X - \mu_X) = \mu + \frac{\sigma^2}{\sigma_X^2}(X - \mu) = X. \]

Alternative Solution:
where the independence of $X, W$ has been used.

(c) For $Z$ in the part (b) compute $P(Z \geq \mu)$.

Solution: $Z \sim \mathcal{N}(\mu, \sigma^2 + 4)$. Therefore,
\[ P(Z \geq \mu) = P\left(\frac{Z - \mu}{\sigma + 4} \geq \frac{\mu - \mu}{\sqrt{\sigma^2 + 4}}\right) = P(\tilde{Z} \geq 0) = Q(0) = \frac{1}{2}, \]
where $\tilde{Z} \sim \mathcal{N}(0, 1)$. 