

ECE 313: Hour Exam II

Wednesday, November 14, 2018

8:45 p.m. — 10:00 p.m.

1. [10 points] Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1$ and $\sigma^2 = 4$. Find $P(|X| < 1)$. Express your solution in terms of the Q function.

Solution:

$$\begin{aligned} P(|X| < 1) &= P(-1 < X < 1) = P(X < 1) - P(X < -1) = \\ &= P\left(\frac{X - \mu}{\sigma} < \frac{1 - 1}{2}\right) - P\left(\frac{X - \mu}{\sigma} < \frac{-1 - 1}{2}\right) \\ &= P(Z < 0) - P(Z < -1), \end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$. Using the symmetry properties of the Q function and the given hint:

$$P(|X| < 1) = Q(0) - Q(1) = \frac{1}{2} - Q(1).$$

2. [6+8+6 points] The three parts are unrelated.

- (a) A random variable T has the exponential distribution with $\lambda = 2$. Find $E(T|T > 1)$.

Solution: Using the memoryless property of exponential distribution,

$$E(T|T > 1) = 1 + E(T) = 1 + 0.5 = 1.5$$

- (b) A random variable X has the uniform distribution on $[0, 2]$. Find $E(X|X > 1)$.

Solution: The conditional distribution of X given $X > 1$ is uniform on the interval $[1, 2]$. Hence

$$E(X|X > 1) = \frac{1+2}{2} = 1.5$$

- (c) Buses arrive at a bus stop according to a Poisson process with arrival rate $\lambda = 1$ bus per minute. Bob arrives at the bus stop at 12 noon and wants to catch the next bus. What is the probability that no bus arrives between 12 noon and 12:02pm? You can leave your answer in terms of e , the base of natural logarithm, e.g. $2e^{-1}$.

Solution: There are two ways to solve this problem. Here is the first way. Denote the number of buses arriving within two minutes by N_2 , which has a Poisson distribution with mean $2\lambda = 2$, hence

$$P(N_2 = 0) = e^{-2}.$$

The second way is to denote the time between 12 noon and the arrival of the first bus by T_1 , which has an exponential distribution with rate $\lambda = 1$, hence

$$P(T_1 > 2) = e^{-2\lambda} = e^{-2}.$$

3. [6+10+8+12 points] Suppose random variables X and Y have the joint pdf

$$f_{X,Y}(u, v) = \begin{cases} u + v & \text{if } 0 \leq u \leq 1, 0 \leq v \leq 1 \\ 0 & \text{else.} \end{cases}$$

- (a) Are X and Y independent? Justify your answer.

Solution: No. Even though the support is a product set, the function $u + v$ cannot be factored as a product of a function of u (alone) and a function of v (alone).

- (b) Determine the marginal pdfs of X and Y .

Solution:

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) dv = \begin{cases} \int_0^1 (u + v) dv = u + \frac{1}{2} & \text{if } 0 \leq u \leq 1 \\ 0 & \text{else.} \end{cases}$$

By symmetry

$$f_Y(v) = \begin{cases} v + \frac{1}{2} & \text{if } 0 \leq v \leq 1 \\ 0 & \text{else.} \end{cases}$$

- (c) Find the conditional pdf $f_{Y|X}(v|u)$ for all values of u for which it is well-defined. Also, be sure to indicate the values of v for which $f_{Y|X}(v|u)$ is zero.

Solution: The pdf $f_X(u) \neq 0$ if and only if $0 \leq u \leq 1$. Thus the conditional pdf $f_{Y|X}(v|u)$ is defined only for $0 \leq u \leq 1$, in which case

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u, v)}{f_X(u)} = \begin{cases} \frac{u+v}{u+\frac{1}{2}} & \text{if } 0 \leq v \leq 1 \\ 0 & \text{else.} \end{cases}$$

- (d) Find $P\{Y > X\}$.

Solution:

$$\begin{aligned} P\{Y > X\} &= \int_{u=0}^1 \int_{v=u}^1 (u + v) dv du = \int_0^1 \left(uv + \frac{v^2}{2} \right) \Big|_u^1 du \\ &= \int_0^1 \left(u + \frac{1}{2} - \frac{3u^2}{2} \right) du = \frac{u}{2} + \frac{u^2}{2} - \frac{u^3}{2} \Big|_0^1 = \frac{1}{2} \end{aligned}$$

We could have guessed this answer by the symmetry in the joint pdf.

4. [8+12 points] Suppose $Y = e^X$, where X is an exponentially distributed random variable with parameter $\lambda > 1$. For both parts, express your answer in terms of λ .

- (a) Find $E[Y]$.

Solution:

$$E[Y] = \int_0^{\infty} e^t \lambda e^{-\lambda t} dt = \int_0^{\infty} \lambda e^{(1-\lambda)t} dt = \frac{\lambda}{1-\lambda} e^{(1-\lambda)t} \Big|_0^{\infty} = \frac{\lambda}{\lambda-1}$$

- (b) Find the CDF $F_Y(v)$.

Solution: Since X is an exponentially distributed random variable with parameter λ , the CDF of X is

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Since $Y = e^X$ and the support for X is $[0, \infty)$, the support for Y is $[1, \infty)$. Therefore, when $v < 1$, $F_Y(v) = 0$. When $v \geq 1$,

$$F_Y(v) = P(Y \leq v) = P(X \leq \ln v) = F_X(\ln v) = 1 - e^{-\lambda \ln v} = 1 - v^{-\lambda}$$

5. [14 points] Consider a binary hypothesis testing problem where

$$\begin{aligned} H_0 : f_X(x) &= \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty \\ H_1 : f_X(x) &= e^{-2|x|}, \quad -\infty < x < \infty \end{aligned} .$$

Find the ML Rule and the corresponding $P_{false\ alarm}$ and P_{miss} .

Solution: The likelihood ratio for this problem is

$$\Lambda(x) = \frac{e^{-2|x|}}{\frac{1}{2}e^{-|x|}} = 2e^{-|x|}.$$

For the ML rule, we declare H_1 when $\Lambda(x) > 1$, i.e.,

$$2e^{-|x|} > 1 \quad \text{or} \quad |x| < -\ln(1/2) = \ln(2) = 0.693.$$

Moreover,

$$P_{false\ alarm} = P(H_1|H_0) = \int_{-\ln(2)}^{\ln(2)} \frac{1}{2}e^{-|x|} dx = 2 \int_0^{\ln(2)} \frac{1}{2}e^{-x} dx = \frac{1}{2}.$$

$$P_{miss} = P(H_0|H_1) = 1 - P(H_1|H_1) = 1 - \int_{-\ln(2)}^{\ln(2)} e^{-2|x|} dx = 1 - 2 \int_0^{\ln(2)} e^{-2x} dx = 1 - \frac{3}{4} = \frac{1}{4}.$$