University of Illinois Fall 2018

ECE 313: Hour Exam I

Wednesday, October 10, 2018
8:45 p.m. — 10:00 p.m.

1. **[8+3+5 points]** Two fair dice are rolled. Define the events:

$$A = \{\text{"sum of the numbers on the dice is odd"}\}$$
$$B = \{\text{"the first die has the number 1"}\}$$

(a) Display the outcomes in a 2-event Karnaugh map corresponding to the events A and B
Solution:

	B^c	B	
	(2,2), (2,4), (2,6), (3,1), (3,3) (3,5), (4,2), (4,4), (4,6), (5,1) (5,3), (5,5), (6,2), (6,4), (6,6)	(1,1), (1,3), (1,5)	A^c
	(2,1), (2,3), (2,5), (3,2), (3,4) (3,6), (4,1), (4,3), (4,5), (5,2) (5,4), (5,6), (6,1), (6,3), (6,5)	(1,2), (1,4) , (1,6)	A

(b) Find $P(AB)$.
Solution: From the Karnaugh map:

$$P(AB) = \frac{3}{36} = \frac{1}{12}.$$

(c) Are the events A and B mutually independent? Explain.
Solution: From the Karnaugh map:

$$P(A) = \frac{18}{36} = \frac{1}{2}, \qquad P(B) = \frac{6}{36} = \frac{1}{6}.$$

Thus $P(AB) = P(A)P(B)$, which means that A and B are independent.

2. **[10+10 points]** Suppose five cards are drawn from a standard 52 card deck of playing cards, with all possibilities being equally likely. (A standard card deck has four suites and each suite has numbers 1 to 13.) A "4 of a kind" is the event that four of the five cards have the same number. For the following questions, you may leave your answer in terms of binomial coefficients (without simplification).

(a) What is the probability of "4 of a kind"?
Solution: There are 13 ways to choose the number for the "4 of a kind", and 12 choices for the number of the remaining card and 4 choices for the suit of that card. Therefore:

$$P(\text{"4 of a kind"}) = \frac{13 \times 12 \times 4}{\binom{52}{5}} = \frac{12 \times 5}{\binom{51}{4}} = \frac{60}{\binom{51}{4}}.$$

(b) What is the conditional probability of "4 of a kind", given that one of the five cards drawn is the Ace of Clubs?

Solution: Given that one of the five cards drawn is 1C, there are $\binom{51}{4}$ possibilities for the remaining 4 cards. There are two ways in which we can form the "4 of a kind": (1) Aces form the "4 of a kind", in which case there are 12 choices for the number of the remaining card and 4 choices for the suit of that card; (2) some other number forms the "4 of a kind", in which case there are 12 choices for that number. Adding up the possibilities we get:

$$P(\text{"4 of a kind"} \mid \text{"one of the cards is 1C"}) = \frac{12 \times 4 + 12}{\binom{51}{4}} = \frac{60}{\binom{51}{4}}.$$

Note that this answer is the same as that of part (a). This is because the conditional probability of "4 of a kind" is the same no matter what the revealed card is.

3. [**6+8+10+10 points**] We have a bag of coins. Pick one coin from the bag and flip the same coin repeatedly.

(a) The coin shows heads with probability p each time it is flipped. We flip the coin n times and denote the number of heads we observe by X. We will use $\hat{p} = \frac{X}{n}$ to estimate p. If we want to estimate p to within 0.1 with 99% confidence, how many times do we need to flip the coin?

Solution: Let n be the number of flips and X be the total number of heads showing up. We can estimate p using $\hat{p} = \frac{X}{n}$. Using the result on confident interval for binomial distribution, we obtain that

$$\begin{aligned} 1 - \frac{1}{a^2} &= 99\% \\ a &= 10 \end{aligned}$$

The half-width of the confidence interval is $\frac{a}{2\sqrt{n}} = \frac{5}{\sqrt{n}}$, which should be less than or equal to 0.1. This requires $n \geq (\frac{5}{0.1})^2 = 2500$.

(b) The coin shows heads with probability p each time it is flipped. We observe the second head at the sixth trial. Compute the ML estimate of p. Show your work.

Solution: Let S_2 be the number of trials until the second head shows. S_2 has a negative binomial distribution with parameter p and $r = 2$. Hence, $P(S_2 = 6) = \binom{5}{1} p^2 (1-p)^4$. Differentiate $P(S_2 = 6)$ with respective to p and set the derivative to 0, we obtain

$$\begin{aligned} 2p(1-p)^4 - 4p^2(1-p)^3 &= 0 \\ 2(1-p) - 4p &= 0 \\ p = \frac{1}{3} \end{aligned}$$

Note that the ML estimate of p is the same if we observe a *total* of two heads out of six trials, regardless of at which trial a head shows up.

(c) We flip the same coin three times and observe the total number of heads, X. Let H_0 be the hypothesis that the coin is fair. Let H_1 be the hypothesis that the coin is biased and shows heads with probability $\frac{1}{4}$. Write down the ML decision rule. Compute p_{miss}.

Solution:

X	0	1	2	3
H_1	$\underline{(\frac{3}{4})^3}$	$\underline{3(\frac{1}{4})(\frac{3}{4})^2}$	$3(\frac{1}{4})^2(\frac{3}{4})$	$(\frac{1}{4})^3$
H_0	$(\frac{1}{2})^3$	$3(\frac{1}{2})^3$	$\underline{3(\frac{1}{2})^3}$	$\underline{(\frac{1}{2})^3}$

$$p_{miss} = P(\text{declare } H_0|H_1) = 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^3 = \frac{5}{32}.$$

(d) Suppose we know that 80% of the coins are fair and 20% of the coins show heads with probability $\frac{1}{3}$. If the first head shows at the second flip, what is the probability that we picked a fair coin?

Solution: Let A be the event that the first head shows at the second flip. Let B be the event that we picked a fair coin.

$$\begin{aligned} P(B|A) &= \frac{P(AB)}{P(A)} = \frac{P(AB)}{P(AB)+P(AB^c)} \\ &= \frac{\frac{4}{5}\times\frac{1}{2}\times\frac{1}{2}}{\frac{4}{5}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{5}\times\frac{2}{3}\times\frac{1}{3}} \\ &= \frac{\frac{1}{5}}{\frac{11}{45}} = \frac{9}{11}. \end{aligned}$$

4. [**6+6 points**] Suppose A, B, and C are events for a probability experiment such that A and B are mutually independent, $P(A) = P(B) = P(C) = 0.4$, $P(AC) = 0.3$, $P(BC) = 0.2$, and $P(ABC) = 0.1$.

(a) What is $P(AB^cC)$?

Solution:

$$P(AB^cC) = P(AC) - P(ABC) = 0.3 - 0.1 = 0.2$$

(b) What is $P(C|AB)$?

Solution:

$$P(C|AB) = \frac{P(ABC)}{P(AB)} = \frac{P(ABC)}{P(A)P(B)} = \frac{0.1}{0.4\times 0.4} = \frac{0.1}{0.16} = \frac{5}{8}.$$

5. [**8+10 points**] The two parts are unrelated.

(a) Alice has a biased coin that shows heads with probability p_1. Bob has another biased coin that shows heads with probability p_2. For each trial, Alice and Bob toss their respective coins at the same time. What is the expected number of trials until the two coins show the same outcome?

Solution: Same outcomes correspond to the event $A = \{HH, TT\}$. Therefore, the probability of getting the same outcome in this experiment is

$$P(A) = p_1p_2 + (1-p_1)(1-p_2).$$

The number of trials until both coins show the same outcome is geometrically distributed with parameter $p = P(A)$. The expected number of trials until the two coins show the same outcome is

$$\frac{1}{P(A)} = \frac{1}{p_1p_2 + (1-p_1)(1-p_2)}.$$

(b) Suppose you play a sequence of N independent games in a casino, where $N \geq 2$ is fixed. The probability that you will win the kth game is $\frac{1}{k}$ for $k = 1, 2, \ldots, N$. You receive one dollar each time you win two games in a row (for example, if you win the first three games, you receive two dollars). What is the expected value of the total amount you will receive? **Hint:** For any integer $n \neq 0, 1$: $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$.

Solution: For $2 \leq k \leq N$, let X_k be the amount of dollars you gain in the kth game, i.e., $X_k \in \{0, 1\}$. The total amount you will gain in the end is $X = \sum_{k=2}^{N} X_k$. Therefore,

$$E[X] = \sum_{k=2}^{N} E[X_k].$$

Moreover,

$$P(X_k = 1) = \frac{1}{k(k-1)}, k = 2, 3, \ldots, N.$$

Since X_k are Bernoulli random variables, $E[X_k] = P(X_k)$ and hence,

$$E[X] = \sum_{k=2}^{N} E[X_k] = \sum_{k=2}^{N} \frac{1}{k(k-1)} = \sum_{k=2}^{N} \left[\frac{1}{(k-1)} - \frac{1}{k} \right] = 1 - \frac{1}{N} = \frac{N-1}{N}.$$