University of Illinois Fall 2017

ECE 313: Hour Exam I

Wednesday, October 11, 2017
8:45 p.m. — 10:00 p.m.

1. **[6 points]** Suppose that a fair coin is independently tossed twice. Define the events:

$$A = \{\text{“The first toss is a head”}\}$$
$$B = \{\text{“The second toss is a head”}\}$$
$$C = \{\text{“Both tosses yield the same outcome”}\}.$$

Are A, B, C pairwise independent? Please show your work.

Solution: The sample space is:

$$\Omega = \{HH, HT, TH, TT\}.$$

Also, $A = \{HH, HT\}$, $B = \{HH, TH\}$ and $C = \{HH, TT\}$. Since the coin is fair:

$$P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}.$$

We have: $A \cap B = \{HH\}$, $A \cap C = \{HH\}$ and $B \cap C = \{HH\}$. Therefore,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4} = P(A)P(B) = P(A)P(C) = P(B)P(C),$$

i.e., A, B, C are pairwise independent.

2. **[10 points]** Suppose a coin is bent or fair with equal probability. In particular, let the probability of showing up head be p, then $P(p = 0.5) = P(p = 0.8) = 0.5$. We toss the coin twice.

(a) Find the probability of getting two heads.
Solution:

$$P(2\text{ heads}) = 0.5 \times 0.5 \times 0.5 + 0.5 \times 0.8 \times 0.8 = 0.445.$$

(b) We will decide if the coin is fair based on the number of tails out of two tosses. How many different decision rules are there? (Hint: The decision rule does not need to be MAP or ML)
Solution: There are a total of three outcomes, 0, 1 or 2. Hence $2^3 = 8$ different decision rules.

3. **[6 points]** A machine produces computer chips that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced chips get passed through a quality control equipment, which is able to detect any chip that is obviously defective and discard it. What is the probability that a chip is good given that it made it through the quality control equipment?

Solution: Define the events:

$$G = \{\text{“Good Chip”}\}$$
$$SD = \{\text{“Slightly Defective Chip”}\}$$
$$OD = \{\text{“Obviously Defective Chip”}\}.$$

By assumption,

$$P(G) = 0.9, \quad P(SD) = 0.02, \quad P(OD) = 0.08.$$

The probability that a chip is *good*, given that it passed the quality control, i.e., it is *obviously not defective* is

$$P(G|OD^c) = \frac{P(G \cap OD^c)}{P(OD^c)} = \frac{P(G)}{1 - P(OD)} = \frac{0.9}{1 - 0.08} = \frac{45}{46}.$$

4. **[12 points]** Consider a computational creativity system like MasterProbo that can generate an infinite sequence of questions. A student uses the system and answers questions correctly with probability 4/5, independently of all other questions.

   (a) What is the expected number of questions the student will get correct before he/she has gotten three questions incorrect?

   Solution: Note that here we are only concerned with the mean, rather than evaluation of the pmf. Let us think of incorrect questions as success with probability 1/5, and so in considering the cumulative trials until getting three questions incorrect, we are in a situation described by a negative binomial random variable with parameters $r = 3$ and $p = 1/5$. The mean is r/p, but we must subtract r from this to get the expected number of correct questions:

$$\frac{r}{p} - r = 3 \cdot 5 - 3 = 12.$$

   (b) What is the probability the first time a student gets a question correct is the fifth question?

   Solution: The time till the first success is governed by a geometric random variable, here with parameter $p = 4/5$, evaluated at $k = 5$:

$$(1/5)^4(4/5) = \frac{4}{5^5} = \frac{4}{3125}.$$

5. **[12 points]** *Ziziphus mauritiana*, also known as Chinese date, ber, Chinee apple, jujube, Indian plum, Regi pandu, and Indian jujube is a tropical fruit tree species belonging to the family *Rhamnaceae*. Some of its fruits are sweet and some are bitter (which we assume to be independent of one another); it is very difficult to tell without tasting. The probability of sweet is p and the probability of bitter is $1 - p$. A forest dweller eats a very large sequence of fruits.

   (a) What is the probability that the number of fruits the forest dweller must eat to get three sweet ones is seven?

   Solution: In a Bernoulli process, the cumulative numbers of trials needed for $r = 3$ successes is governed by the negative binomial distribution, here with success probability p, evaluated at $n = 7$:

$$\binom{6}{2}(p)^3(1-p)^4.$$

   (b) If the forest dweller decides to stop at k fruits, what is the probability the last one will be sweet?

   Solution: Each draw in a Bernoulli process is independent and identically distributed. Hence, the fruit drawn at any given time is just governed by a Bernoulli random variable, with success probability p. Thus, the probability the last one is sweet is just p.

6. **[16 points]** Consider two hypotheses, H_1 and H_0. If H_1 is true, the observation X has the distribution $P(X=1)=0.2$ and $P(X=2)=0.8$. If H_0 is true, $P(X=1)=P(X=2)=0.5$.

   (a) Specify the ML decision rule given the observation X. What is the miss probability?
   Solution:

	$X=1$	$X=2$
H_1	0.2	0.8
H_0	0.5	0.5
Λ	0.4	1.6

   For ML decision rule, the threshold is 1. Hence if $X=1$, declare H_0. Otherwise, declare H_1.

$$p_{\text{miss}} = P(\text{declare } H_0 | H_1 \text{ is true}) = P(X=1|H_1 \text{ is true}) = 0.2.$$

   (b) Given the prior distribution where H_1 is true with probability $\pi_1 = 0.8$. Specify the MAP decision rule given the observation X. What is the miss probability?
   Solution: The MAP threshold for the likelihood ratio is $\pi_0/\pi_1 = 0.25$. Hence we always declare H_1.

$$p_{\text{miss}} = P(\text{declare } H_0 | H_1 \text{ is true}) = 0.$$

   (c) Given the same prior distribution as in part (b), suppose we have two independent observation of X, what is the new MAP decision rule based on two observations?
   Solution:

	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
H_1	0.04	0.16	0.16	0.64
H_0	0.25	0.25	0.25	0.25
Λ	0.16	0.64	0.64	2.56

   The MAP threshold remains unchanged at 0.25. Hence we declare H_0 when we observe $(1,1)$, otherwise we declare H_1.

7. **[10 points]** Three prisoners A, B and C are sentenced to death. The governor, however, has selected one of them to be pardoned, each with equal probability. The warden knows who is to be pardoned, but is not allowed to tell. Instead the warden agrees to tell A one name that is not to be pardoned as follows.

   - If B is to be pardoned, the warden gives C's name.
   - If C is to be pardoned, the warden gives B's name.
   - If A is to be pardoned, the warden gives B or C's name with equal probability.

   Find the conditional probability that A is to be pardoned given the warden gives B's name.
   Solution: By Bayes' rule,

$$P(\text{A pardoned} \mid \text{B's name given}) = \frac{P(\text{B's name given} \mid \text{A pardoned})P(\text{A pardoned})}{P(\text{B's name given})}.$$

   We have $P(\text{B's name given} \mid \text{A pardoned}) = \frac{1}{2}$ and $P(\text{A pardoned}) = \frac{1}{3}$. By symmetry we have $P(\text{B's name given}) = \frac{1}{2}$. So $P(\text{B's name given} \mid \text{A pardoned}) = \frac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$.

8. **[16 points]** In basketball games players make three-point field goal attempts and some of the attempts result in successful three-point field goals. Assume each of the attempts of a player is successful independently with equal probability called the three-pointer shooting percentage. Different players may have different three-pointer shooting percentages.

   (a) Alice has a three-pointer shooting percentage of 0.4 and successfully made one three-point field goal in a game. The number of three-point field goal attempts by Alice in this game is known to be less than or equal to three, but otherwise unknown. Give a maximum likelihood estimate of the number of her three-point field goal attempts.

   Solution: The number of successfully made 3-point field goals by Alice follows a binomial distribution with parameters n and p, where n is the number of attempts and $p = 0.4$ is the shooting percentage. The likelihood of successfully making one out of n attempts is

$$\begin{aligned} L(n) &= \binom{n}{1} p^1 (1-p)^{n-1} \\ &= np(1-p)^{n-1}. \end{aligned}$$

   So the likelihoods for $n = 1, 2, 3$ are

$$\begin{aligned} L(1) &= p = 0.4. \\ L(2) &= 2p(1-p) = 0.48. \\ L(3) &= 3p(1-p)^2 = 0.432. \end{aligned}$$

   Then $\hat{n}_{\text{ML}} = 2$ is the ML estimate of the number of 3-point field goal attempts by Alice in the game.

   (b) Let Bob's three-pointer shooting percentage p be unknown. For practice Bob attempts n three-point field goals and successfully makes X of them. Let his three-pointer shooting percentage be estimated by $\hat{p} = \frac{X}{n}$. If p is to be estimated within 0.05 (i.e., by an interval estimate of length 0.1) with 75% confidence, find the minimum value of n based on

$$P\left\{ p \in \left(\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}} \right) \right\} \geq 1 - \frac{1}{a^2},$$

   where a can be any positive number.

   Solution: For 75% confidence a should be 2. Then by setting $\frac{a}{2\sqrt{n}} = 0.05$ we get $n = 400$.

9. **[12 points]** Suppose you want to divide a 52 card deck into four hands with 13 cards each. What is the probability that each hand has a king? (Hint: All 52 cards are distinct and there are only four kings in the set of the cards)

   Solution: Let H_i be the event that the i^{th} hand has one king. We have the conditional probabilities

$$P(H_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}; P(H_2|H_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}; P(H_3|H_1H_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}; P(H_4|H_1H_2H_3) = 1$$

$$\begin{aligned} P(H_1H_2H_3H_4) &= P(H_4|H_1H_2H_3)P(H_3|H_1H_2)P(H_2|H_1) \\ &= \frac{4!\binom{48}{12}\binom{36}{12}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} \\ &= \frac{48!4!13^4}{52!} = \frac{13^3}{17 \times 25 \times 49} = \frac{2197}{20825}. \end{aligned}$$