ECE 313: Final Exam
Monday, December 12, 2016
7 p.m. — 10 p.m.
Aa-Fh in room ECEB 1013
Fi-Zz in room ECEB 1002

1. [14 points] A drawer contains 4 black, 6 red, and 8 yellow socks. Two socks are selected at random from the drawer.

(a) What is the probability the two socks are of the same color?

**Solution:** Let $B$, $R$, and $Y$ denote the sets of black, red, and yellow socks, with cardinalities 4, 6, and 8, respectively. A suitable choice of sample space for this experiment is

$$\Omega = \{S : |S| = 2 \text{ and } S \subseteq B \cup R \cup Y\},$$

where $S$ represents the set of two socks selected. The cardinality of $\Omega$ is

$$|\Omega| = \binom{4 + 6 + 8}{2} = \binom{18}{2} = 153.$$

Let $F$ be the event $F = \{S : |S| = 2 \text{ and } S \subseteq B \text{ or } S \subseteq R \text{ or } S \subseteq Y\}$. Then,

$$|F| = \binom{4}{2} + \binom{6}{2} + \binom{8}{2} = 6 + 15 + 28 = 49.$$

Thus,

$$P(F) = \frac{49}{153}.$$

(b) What is the conditional probability both socks are yellow given they are of the same color?

**Solution:** Let $G = \{S : |S| = 2 \text{ and } S \subseteq Y\}$. Note that $G \subseteq F$ and $|G| = \binom{8}{2} = 28$. Therefore,

$$P(G|F) = \frac{P(GF)}{P(F)} = \frac{P(G)}{P(F)} = \frac{28}{49} = \frac{4}{7}. $$

2. [14 points] The two parts of this problem are unrelated.

(a) Suppose $A$, $B$, and $C$ are events for a probability experiment such that $B$ and $C$ are mutually independent, $P(A) = P(B^c) = P(C) = 0.5$, $P(AB) = P(AC) = 0.3$, and $P(ABC) = 0.1$. Fill in the probabilities of all events in a Karnaugh map. Show your work AND use the map on the right to depict your final answer.

**Solution:** Start by filling in $P(ABC) = 0.1$. Then use $P(AC) = 0.3$ to get $P(AB^cC) = P(AC) - P(ABC) = 0.3 - 0.1 = 0.2$. Similarly, use $P(AB) = 0.3$ to
get \( P(ABC^c) = P(AB) - P(ABC) = 0.3 - 0.1 = 0.2 \). The independence of \( B \) and \( C \) and the given probabilities of \( B \) and \( C \) yield \( P(BC) = P(B)P(C) = 0.25 \), from which we conclude as before that \( P(A^cBC^c) = P(BC) - P(ABC) = 0.25 - 0.1 = 0.15 \). Use \( P(A) = 0.5 \) to get \( P(AB^cC^c) = 0 \); use \( P(B) = 0.5 \) to get \( P(A^cBC^c) = 0.05 \); use \( P(C) = 0.5 \) to get \( PA^cB^cC^c) = 0.05 \). Finally, all probabilities add to one;

\[
P(A^cB^cC^c) = 0.25.
\]

(b) Let \( A, B \) be two disjoint events on a sample space \( \Omega \). Find a formula for the probability of \( A \) occurring before \( B \) in an infinite sequence of independent trials.

**Solution:** \( A \) and \( B \) occur with probabilities \( P(A) \) and \( P(B) \), respectively. Consider the first trial:

**First Trial:** Either \( A \) occurs or \( B \) occurs or neither \( A \) nor \( B \) occurs.
- If \( A \) occurs, then the probability that \( A \) occurs before \( B \) is 1.
- If \( B \) occurs, then the probability that \( A \) occurs before \( B \) is 0.
- If neither \( A \) nor \( B \) occurs, then the process starts over.

Let \( s \) be the probability that neither \( A \) nor \( B \) occurs in a given independent trial. Then \( s = 1 - P(A) - P(B) \) due to \( A \cap B = \emptyset \). Therefore,

\[
P(A \text{ before } B) = P(A) + sP(A) + s^2P(A) + \cdots + s^nP(A) + \cdots = P(A) \sum_{n=0}^{\infty} s^n = P(A) \frac{1}{1-s} = \frac{P(A)}{P(A) + P(B)}.
\]

**Alternatively:** Let \( s \) be the probability that neither \( A \) nor \( B \) occurs in a given independent trial. If neither \( A \) nor \( B \) occurs on the first trial, then the process starts over. So \( P(A \text{ before } B) = P(A) + sP(A \text{ before } B) \). Solving this equation for \( P(A \text{ before } B) \) yields \( P(A \text{ before } B) = \frac{P(A)}{1-s} = \frac{P(A)}{P(A) + P(B)} \).

3. **[20 points]** Suppose two teams, Cubs and Indians, play a best-of-seven series of games. Assume that games are independent, that ties are not possible in each game, and that Cubs wins a given game with probability \( p \in (0, 1) \). The series ends as soon as one of the teams has won four games. Let \( G \) denote the total number of games played.

(a) Obtain the probability that Cubs win exactly 2 of the first 4 games.

**Solution:** The number of games that Cubs win out of the first 4 games is \( \text{Binomial}(4, p) \), hence

\[
P\{\text{Cubs win exactly 2 of the first 4 games}\} = \binom{4}{2}p^2(1-p)^2
\]
(b) What is the expected number of games that Cubs will win out of the first 4 games? 
\textbf{Solution:} From part (a), the number of games that Cubs win out of the first 4 games is $\text{Binomial}(4, p)$, hence

$$E[\text{number of games that Cubs will win out of the first 4 games}] = 4p$$

(c) Obtain the probability $P\{G = 6, \text{Cubs win the series}\}$.
\textbf{Solution:} Let $W_C = \{\text{Cubs win the series}\}$. For event $\{G = 6, W_C\}$ to occur we need Cubs to win 3 out of the first 5 games, and Cubs must win the 6th game. The number of games that Cubs win out of the first 5 games is $\text{Binomial}(5, p)$, and $p$ is the probability that Cubs win game 6 if it is reached. Hence,

$$P\{G = 6, W_C\} = \left(\binom{5}{3} p^3 (1-p)^2\right) p = \binom{5}{3} p^4 (1-p)^2$$

(d) Obtain $p_G(n)$, the pmf of $G$, for all $n$.
\textbf{Solution:} Clearly $G \in \{4, 5, 6, 7\}$. Using total probability, and following reasoning similar to part (b), for $n \in \{4, 5, 6, 7\}$:

$$p_G(n) = P\{G = n\} = P\{G = n, \text{Cubs win}\} + P\{G = n, \text{Cubs do not win}\}$$

$$= \left(\binom{n-1}{3} p^3 (1-p)^{n-1-3}\right) p + \left(\binom{n-1}{3} (1-p)^3 p^{n-1-3}\right) (1-p)$$

$$= \binom{n-1}{3} p^4 (1-p)^{n-4} + \binom{n-1}{3} (1-p)^4 p^{n-4}$$

4. [14 points] Suppose $S$ and $T$ represent the lifetimes of two phones, the lifetimes are independent, and each has the exponential distribution with parameter $\lambda = 1$.

(a) Obtain $P\{|S - T| \leq 1\}$.
\textbf{Solution:} $P\{|S - T| \leq 1\} = \int \int_R e^{-u} e^{-v} dudv$, where $R$ is the infinite strip in the positive quadrant defined by $R = \{u \geq 0, v \geq 0, |u - v| \leq 1\}$. The complement of $R$ in the positive quadrant is the union of the region $S_1 = \{u \geq 1, 0 \leq v \leq u - 1\}$ below $R$, and a similar region, $S_2$, above $R$. By symmetry, $P\{(S, T) \in S_1\} = P\{(S, T) \in S_2\}$ so that $P\{|S - T| \leq 1\} = 1 - 2P\{(S, T) \in S_1\}$. Since

$$P\{(S, T) \in S_1\} = \int_0^\infty \int_{v+1}^\infty e^{-u} e^{-v} dudv$$

$$= \int_0^\infty e^{-v} \int_{v+1}^\infty e^{-u} dudv$$

$$= \int_0^\infty e^{-2v} dv = \frac{e^{-1}}{2},$$

it follows that $P\{|S - T| \leq 1\} = 1 - e^{-1}$.

\text{ALTERNATIVELY,} $|S - T|$ is the remaining lifetime of the other phone, after one phone fails. By the memoryless property of the exponential distribution, it follows that $|S - T|$ has the same distribution as $S$ or $T$. So $P\{|S - T| \leq 1\} = 1 - e^{-1}$. 

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(b) Let \( Z = (S - 1)^2 \). Obtain \( f_Z(c) \), the pdf of \( Z \), for all \( c \).

**Solution:** Clearly \( P\{Z \geq 0\} = 1 \). For \( c \geq 0 \), \( F_Z(c) = P\{(S - 1)^2 \leq c\} = P\{-\sqrt{c} \leq S - 1 \leq \sqrt{c}\} = P\{1 - \sqrt{c} \leq S \leq 1 + \sqrt{c}\} \). So

\[
F_Z(c) = \begin{cases} 
0 & c < 0 \\
\int_{\sqrt{c}}^{1+\sqrt{c}} e^{-u} du = e^{-1+\sqrt{c}} - e^{-1-\sqrt{c}} & 0 \leq c < 1 \\
\int_0^{1+\sqrt{c}} e^{-u} du = 1 - e^{-1-\sqrt{c}} & c \geq 1 
\end{cases}
\]

Differentiating with respect to \( c \) yields

\[
f_Z(c) = \begin{cases} 
0 & c < 0 \\
\frac{e^{-1+\sqrt{c}} - e^{-1-\sqrt{c}}}{2\sqrt{c}} & 0 \leq c < 1 \\
\frac{e^{-1-\sqrt{c}}}{2\sqrt{c}} & c \geq 1 
\end{cases}
\]

5. **[20 points]** Assume power surges occur as a Poisson process with rate 3 per hour. These events cause damage to a certain system (say, a computer).

   (a) Obtain \( F_{T_3}(t) \), the CDF of the time when the third power surge occurs, for all \( t \geq 0 \), measured for some reference time 0. NOTE: Give a simple answer that does not involve an integral or the sum of an infinite series. (**Hint:** It might be easier to first obtain the complementary CDF.)

   **Solution:** The third surge takes place by time \( t \) if and only if at least three surges occur by time \( t \). That is, \( T_3 \leq t \) if and only if \( N_t \geq 3 \). Thus,

   \[ P\{T_3 \leq t\} = P\{N_t \geq 3\} = 1 - P\{N_t \leq 2\} = 1 - (1 + \lambda t + \frac{(\lambda t)^2}{2})e^{-\lambda t}, \text{ where } \lambda = \frac{1}{6}. \]

   (b) Assume that a single power surge occurring in a certain 10 minute period will cause the system to crash. What is the probability that the system will crash in that period?

   **Solution:** The rate of power surges is \( \lambda = 3 \) per hour. The duration of the service period, \( t_o \), is 10 minutes, or \( t_o = 1/6 \) hour, and \( \lambda t_o = 1/2 \). Let the number of power surges in 10 minutes be \( N \).

   \[ P\{N \geq 1\} = 1 - P\{N = 0\} = 1 - e^{-1/2}. \]

(c) Obtain

\( P\{\text{exactly 1 power surge during 1-3pm AND exactly 2 power surges during 2-6pm}\} \).

**Solution:** The two time intervals overlap, so we need to look at the time intervals \( I_1 = [1, 2], I_2 = (2, 3], \) and \( I_3 = (3, 6] \). We want to find the probability of one power surge during \( I_1 \cup I_2 \) and two power surges during \( I_2 \cup I_3 \). There are two mutually exclusive ways for this to happen:

- (one surge in \( I_1 \), no surges in \( I_2 \), two surges in \( I_3 \))
- (no surge in \( I_1 \), one surge in \( I_2 \), one surge in \( I_3 \)).
These two events have probabilities \((\lambda e^{-\lambda}) (e^{-\lambda}) \left( \frac{(3\lambda)^2 e^{-3\lambda}}{2} \right)\) and \((e^{-\lambda}) (\lambda e^{-\lambda})(3\lambda e^{-3\lambda})\), respectively. Adding these gives the total probability, \[ \frac{2\lambda^3}{9} + 3\lambda^2 \] \( e^{-5\lambda} = (148.5) e^{-15}. \]

6. [22 points] Let \((X, Y)\) be uniformly distributed over the triangular region with vertices \((0, 0)\), \((1/2, 2)\), and \((1, 0)\).

(a) Obtain \(f_{X,Y}(u, v)\), the joint pdf of \(X\) and \(Y\), for all \(u\) and \(v\).

**Solution:** The triangle has height 2 and base one, so it has unit area, so the joint pdf is one inside the triangle and zero outside. That is,

\[
f_{X,Y}(u, v) = \begin{cases} 1 & (u, v) \in T \\ 0 & \text{otherwise} \end{cases}
\]

where \(T = \{(u, v); 0 \leq u \leq 1, 0 \leq v \leq \min(4u, 4 - 4u)\}\), or equivalently, \(T = \{(u, v) : 0 \leq v \leq 2, \frac{v}{4} \leq u \leq 1 - \frac{v}{4}\}\).

(b) Obtain \(f_Y(v)\), the marginal pdf of \(Y\), for all \(v\).

**Solution:** For \(v \geq 2\) or \(v < 0\), \(f_Y(v) = 0\). For \(0 \leq v < 2\),

\[
f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) du = \int_{\frac{v}{4}}^{1-\frac{v}{2}} 1 du
\]

\[
= 1 - \frac{v}{2}
\]

(c) Obtain \(f_{X|Y}(u|v)\), the conditional pdf of \(X\) given \(Y\), for all \(u\) and \(v\).

**Solution:** For \(0 \leq v < 2\),

\[
f_{X|Y}(u|v) = \frac{f_{X,Y}(u, v)}{f_Y(v)} = \begin{cases} \frac{2}{2-v} & \frac{v}{4} < u < 1 - \frac{v}{4} \\ 0 & \text{else} \end{cases}
\]

That is, given \(Y = v\), the conditional distribution of \(X\) is uniform over the interval \(\left[\frac{v}{4}, 1 - \frac{v}{4}\right]\). For \(v < 0\) or \(v \geq 2\), the conditional pdf \(f_{X|Y}(u|v)\) is not defined.

(d) Obtain \(E[X|Y = v]\) for all \(v\).

**Solution:** The mean of the uniform distribution over \(\left[\frac{v}{4}, 1 - \frac{v}{4}\right]\) is the midpoint of the interval, or \(\frac{1}{2}\). Thus, for \(0 \leq v < 2\), \(E[X|Y = v] = \frac{1}{2}\). For other \(v\), \(E[X|Y = v]\) is not defined.) Another way to get this result is to use the formulas:

\[
E[X|Y = v] = \int_{-\infty}^{\infty} u f_{X|Y}(u|v) du
\]

\[
= \int_{\frac{v}{4}}^{1-\frac{v}{2}} \frac{2u}{2-v} du
\]

\[
= \frac{1}{2-v} \left. u^{1-\frac{v}{4}} \right|_{\frac{v}{4}}^{1-\frac{v}{2}} = \frac{1}{2}
\]
(e) Determine if $X$ and $Y$ are independent and indicate why or why not.

**Solution:** $X$ and $Y$ are not independent because the support is not a product set. Another reason is that $f_{X|Y}(u|v)$ depends on $v$.

7. **[18 points]** Consider an On-Off Keying (OOK) communication system, where we either transmit $x = 0$ or $x = A$ with $A > 0$ being a constant. At the receiver side, detecting if a “0” was transmitted ($x = 0$) or a “1” was transmitted ($x = A$) can be posed as the following binary hypothesis testing problem for observation $Y$:

$$
\mathcal{H}_0 : Y = W \quad \mathcal{H}_1 : Y = A + W
$$

where $W$ is a $\mathcal{N}(0, \sigma^2)$ random variable corresponding to additive noise at the receiver.

(a) Determine $f_0(y)$, the pdf of $Y$ under $\mathcal{H}_0$, and also $f_1(y)$, the pdf of $Y$ under $\mathcal{H}_1$.

**Solution:** For $\mathcal{H}_0$, $Y = W$ hence

$$f_0(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}.$$

For $\mathcal{H}_1$, $Y = A + W$. Since $Y$ is obtained from $W$ by adding the constant $A$, the pdf of $Y$ is obtained by shifting the pdf of $W$ to the right by $A$. That is, under $\mathcal{H}_1$, $Y$ has the $\mathcal{N}(A, \sigma^2)$ distribution:

$$f_1(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}}.$$

(b) Determine the MAP decision rule assuming the priors $\pi_0$ and $\pi_1$ are known. Express the rule in terms of $Y$ in the simplest way possible.

**Solution:** The likelihood ratio test for the MAP rule is:

$$\frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-A)^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}y^2}} > \frac{\pi_0}{\pi_1}.$$

Cancelling common factors and taking the logarithm to both sides yields:

$$-\frac{1}{2\sigma^2}(-2Ay + A^2) > \ln \left( \frac{\pi_0}{\pi_1} \right).$$

Hence, the MAP rule decides $\mathcal{H}_1$ if $Y > \frac{\sigma^2}{A} \ln \left( \frac{\pi_0}{\pi_1} \right) + \frac{A}{2}$ and $\mathcal{H}_0$ otherwise.

(c) Assume that $\pi_0 = \pi_1$. Determine the average error probability, $p_e$. You can leave your answer in terms of the $Q$ or the $\Phi$ functions.

**Solution:** If $\pi_0 = \pi_1$, the MAP rule decides $\mathcal{H}_1$ if $Y > \frac{A}{2}$ and $\mathcal{H}_0$ otherwise.

$$p_{FA} = P(\text{decide } \mathcal{H}_1 | \mathcal{H}_0 \text{ is true}) = P \left( y > \frac{A}{2} | \mathcal{H}_0 \right) = Q \left( \frac{A}{2\sigma} \right) = Q \left( \frac{A}{2\sigma} \right).$$

$$p_{miss} = P(\text{decide } \mathcal{H}_0 | \mathcal{H}_1 \text{ is true}) = P \left( y \leq \frac{A}{2} | \mathcal{H}_1 \right) = \Phi \left( \frac{A - A}{\sigma} \right) = \Phi \left( -\frac{A}{2\sigma} \right) = Q \left( \frac{A}{2\sigma} \right).$$
Thus,
\[ p_c = \pi_0 p_{F,A} + \pi_1 p_{miss} = \frac{1}{2} Q \left( \frac{A}{2\sigma} \right) + \frac{1}{2} Q \left( \frac{A}{2\sigma} \right) = Q \left( \frac{A}{2\sigma} \right). \]

8. [18 points] Suppose \( X \) and \( Y \) are zero-mean unit-variance jointly Gaussian random variables with correlation coefficient \( \rho = 0.5 \).

(a) Obtain \( \text{Var}(3X - 2Y) \).

**Solution:** \( \text{Var}(3X - 2Y) = 3^2 \cdot \text{Var}(X) + 2^2 \cdot \text{Var}(Y) - 2 \cdot 3 \cdot 2 \cdot \text{cov}(X, Y) = 9 + 4 - 12 \times \frac{1}{2} = 7. \)

(b) Obtain \( P\{ (3X - 2Y)^2 \leq 28 \} \) in terms of the \( Q \) or the \( \Phi \) functions.

**Solution:** \( \mathbb{E}[3X - 2Y] = 3 \cdot \mathbb{E}[X] - 2 \cdot \mathbb{E}[Y] = 0. \) Furthermore, since \( X \) and \( Y \) are *jointly Gaussian* random variables, \( 3X - 2Y \) is also a Gaussian random variable, and we have that
\[ P\{ (3X - 2Y)^2 \leq 28 \} = P\{ -\sqrt{28} \leq 3X - 2Y \leq \sqrt{28} \} = \Phi \left( \frac{\sqrt{28} - 0}{\sqrt{7}} \right) - \Phi \left( -\frac{\sqrt{28} - 0}{\sqrt{7}} \right) = \Phi(2) - \Phi(-2) = \Phi(2) - [1 - \Phi(2)] = 2\Phi(2) - 1. \]

(c) Obtain \( \mathbb{E}[Y \mid X = 3] \).

**Solution:** Since \( X \) and \( Y \) are *jointly Gaussian* random variables, the conditional mean of \( Y \) given \( X = \alpha \) is the same as the linear MMSE estimator of \( X \) given \( X = \alpha \), viz. \( \mu_y + \rho (\sigma_y / \sigma_x) (\alpha - \mu_x) \)
\[ = 0 + 0.5 \times 1 \times (3 - 0) = 3/2. \]

9. [12 points] Observations \( X_1, \ldots, X_T \) produced by a drone’s altimeter are assumed to have the form \( X_t = bt + W_t \) where \( b \) is an unknown constant representing the rate of ascent of the drone (if \( b < 0 \) it means the drone is descending) and \( W_1, \ldots, W_T \) represent observation noise and are assumed to be independent, \( N(0, 1) \) random variables.

(a) Write down the joint pdf of \( X_1, \ldots, X_T \).

**Solution:** \( X_t \) is \( N(bt, 1) \) so \( f_{X_t}(x_t) = \frac{1}{\sqrt{2\pi}} e^{-(x_t - bt)^2 / 2} \). Since the observations are independent, the joint pdf is the product of the marginal pdfs:
\[ f_{X_1, \ldots, X_T}(x_1, \ldots, x_T) = \frac{1}{(2\pi)^{T/2}} e^{-\sum_{t=1}^T \frac{(x_t - bt)^2}{2}} \]

(b) Obtain the maximum likelihood estimator of \( b \) for a particular vector of observations \( x_1, \ldots, x_T \).

**Solution:** \( \hat{b}_{ML} \) is the value of \( b \) that maximizes \( f_{X_1, \ldots, X_T}(x_1, \ldots, x_T) \), or equivalently, minimizes \( \sum_{t=1}^T \frac{(x_t - bt)^2}{2} \). This is a quadratic function of \( b \) that is minimized by setting the derivative to zero.
\[ \frac{d}{db} = \sum_{t=1}^T (x_t - bt)(-t) = b \sum_{t=1}^T t^2 - \sum_{t=1}^T x_t t \]
Setting the derivative to zero yields
\[ \hat{b}_{ML} = \frac{\sum_{t=1}^{T} x_t t}{\sum_{t=1}^{T} t^2}. \]

10. [18 points] Suppose \( U \) and \( V \) are independent random variables such that \( U \) is uniformly distributed over \([0, 1]\) and \( V \) is uniformly distributed over \([0, 2]\). Let \( S = U + V \).

(a) Obtain the mean and variance of \( S \).

**Solution:**
\[ E[S] = E[U] + E[V] = 0.5 + 1 = 1.5. \]
\[ \text{Var}(S) = \text{Var}(U) + \text{Var}(V) = \frac{1}{12} + \frac{2^2}{12} = \frac{5}{12}. \]

(b) Derive and carefully sketch the pdf of \( S \).

**Solution:**
\[ f_S(c) = \int_{-\infty}^{\infty} f_U(u) f_V(c-u) du = \begin{cases} 
  c/2 & 0 \leq c \leq 1 \\
  1/2 & 1 \leq c \leq 2 \\
  (c-2)/2 & 2 \leq c \leq 3 \\
  0 & \text{else} 
\end{cases} \]

(c) Obtain \( \hat{E}[U|S] \), the minimum mean square error linear estimator of \( U \) given \( S \).

**Solution:**
\[ \text{Cov}(U, S) = \text{Cov}(U, U + V) = \text{Var}(U) = \frac{1}{12}. \]
Thus,
\[ \hat{E}[U|S] = E[U] + \frac{\text{Cov}(U, S)}{\text{Var}(S)} (S - E[S]) = \frac{1}{2} + \frac{1}{5} (S - 1.5) \]

11. [30 points] (3 points per answer)

In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose \( X \) and \( Y \) are jointly continuous-type random variables with finite variance.

**TRUE** \hspace{0.5cm} **FALSE**

\[ \square \quad \square \quad \text{If the MMSE for estimating} \ Y \text{from} \ X \text{is} \ \text{Var}(Y), \text{then} \ X \text{and} \ Y \text{must be uncorrelated.} \]

\[ \square \quad \square \quad \text{If} \ X \text{and} \ Y \text{are uncorrelated then the MMSE for estimating} \ Y \text{from} \ X \text{is} \ \text{Var}(Y). \]

\[ \square \quad \square \quad \text{If} \ X \text{and} \ Y \text{are uncorrelated and jointly Gaussian, then} \ \text{the MMSE for estimating} \ Y \text{from} \ X \text{is} \ \text{Var}(Y). \]

**Solution:** True, False, True
(b) Let $X_1, \ldots, X_m$ be independent random variables, each with the binomial distribution with parameters 10 and $p$, where $0 < p < 1$, and let $S_m = X_1 + \ldots + X_m$.

\begin{tabular}{cc}
TRUE & FALSE \\
\square & \square \\
\end{tabular}

$S_m$ has a binomial distribution

\begin{tabular}{cc}
\square & \square \\
\end{tabular}

$\lim_{m \to \infty} P \left\{ \frac{S_m}{m} \geq 10p(1-p) \right\} = 1$

**Solution:** True, True

(c) Consider a binary hypothesis testing problem. Let the subscript $ML$ denote the maximum likelihood rule, and subscript $MAP$ denote the maximum a posteriori rule.

\begin{tabular}{cc}
TRUE & FALSE \\
\square & \square \\
\end{tabular}

It is possible that $p_{\text{miss},ML} < p_{\text{miss},MAP}$.

\begin{tabular}{cc}
\square & \square \\
\end{tabular}

It is possible that $p_{\text{false alarm},ML} = p_{\text{false alarm},MAP}$.

\begin{tabular}{cc}
\square & \square \\
\end{tabular}

If $\pi_0 > \pi_1$ it is possible that $p_{\text{miss},ML} < p_{\text{miss},MAP}$.

**Solution:** True, True, True

(d) Let $X$ and $Y$ be uncorrelated, jointly Gaussian random variables, with parameters $\mu_X$, $\mu_Y$, $\sigma_X^2$ and $\sigma_Y^2$.

\begin{tabular}{cc}
TRUE & FALSE \\
\square & \square \\
\end{tabular}

$f_{XY}(u,v) = f_X(u)f_Y(v)$ for all real $u,v$

\begin{tabular}{cc}
\square & \square \\
\end{tabular}

$E[XY] = 0$.

**Solution:** True, False