University of Illinois Fall 2016

ECE 313: Final Exam

Monday, December 12, 2016
7 p.m. — 10 p.m.
Aa-Fh in room ECEB 1013
Fi-Zz in room ECEB 1002

Name: (in BLOCK CAPITALS) ______________________________

NetID: ______________________________

Signature: ______________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5”×11” sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 11 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 14 points ________
2. 14 points ________
3. 20 points ________
4. 14 points ________
5. 20 points ________
6. 22 points ________
7. 18 points ________
8. 18 points ________
9. 12 points ________
10. 18 points ________
11. 30 points ________

Total (200 points) ________

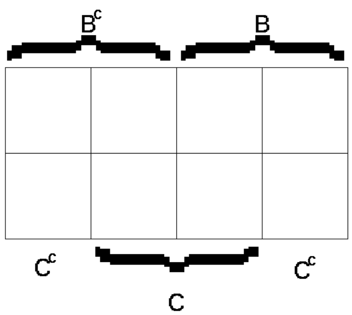
1. [**14 points**] A drawer contains 4 black, 6 red, and 8 yellow socks. Two socks are selected at random from the drawer.

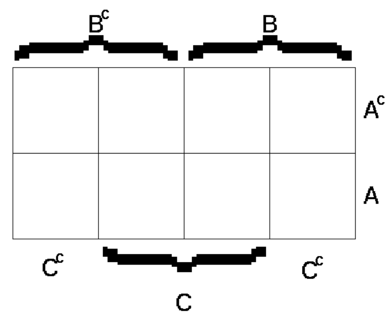
   (a) What is the probability the two socks are of the same color?

   (b) What is the conditional probability both socks are yellow given they are of the same color?

2. [**14 points**] The two parts of this problem are unrelated.

(a) Suppose A, B, and C are events for a probability experiment such that B and C are mutually independent, $P(A) = P(B^c) = P(C) = 0.5$, $P(AB) = P(AC) = 0.3$, and $P(ABC) = 0.1$. Fill in the probabilities of all events in a Karnaugh map. Show your work AND use the map on the right to depict your final answer.





(b) Let A, B be two disjoint events on a sample space Ω. Find a formula for the probability of A occurring before B in an infinite sequence of independent trials.

3. **[20 points]** Suppose two teams, Cubs and Indians, play a best-of-seven series of games. Assume that games are independent, that ties are not possible in each game, and that Cubs wins a given game with probability $p \in (0, 1)$. The series ends as soon as one of the teams has won four games. Let G denote the total number of games played.

(a) Obtain the probability that Cubs win exactly 2 of the first 4 games.

(b) What is the expected number of games that Cubs will win out of the first 4 games?

(c) Obtain the probability $P\{G = 6, \text{Cubs win the series}\}$.

(d) Obtain $p_G(n)$, the pmf of G, for all n.

4. **[14 points]** Suppose S and T represent the lifetimes of two phones, the lifetimes are independent, and each has the exponential distribution with parameter $\lambda = 1$.

   (a) Obtain $P\{|S - T| \leq 1\}$.

   (b) Let $Z = (S - 1)^2$. Obtain $f_Z(c)$, the pdf of Z, for all c.

5. **[20 points]** Assume power surges occur as a Poisson process with rate 3 per hour. These events cause damage to a certain system (say, a computer).

   (a) Obtain $F_{T_3}(t)$, the CDF of the time when the third power surge occurs, for all $t \geq 0$, measured for some reference time 0. NOTE: Give a simple answer that does not involve an integral or the sum of an infinite series. (*Hint:* It might be easier to first obtain the complementary CDF.)

   (b) Assume that a single power surge occurring in a certain 10 minute period will cause the system to crash. What is the probability that the system will crash in that period?

   (c) Obtain
   $P\{$exactly 1 power surge during 1-3pm AND exactly 2 power surges during 2-6pm$\}$.

6. **[22 points]** Let (X, Y) be uniformly distributed over the triangular region with vertices $(0, 0)$, $(1/2, 2)$, and $(1, 0)$.

(a) Obtain $f_{X,Y}(u, v)$, the joint pdf of X and Y, for all u and v.

(b) Obtain $f_Y(v)$, the marginal pdf of Y, for all v.

(c) Obtain $f_{X|Y}(u|v)$, the conditional pdf of X given Y, for all u and v.

(d) Obtain $E[X|Y = v]$ for all v.

(e) Determine if X and Y are independent and indicate why or why not.

7. **[18 points]** Consider an On-Off Keying (OOK) comunication system, where we either transmit $x = 0$ or $x = A$ with $A > 0$ being a constant. At the receiver side, detecting if a "0" was transmitted ($x = 0$) or a "1" was transmitted ($x = A$) can be posed as the following binary hypothesis testing problem for observation Y:

$$\mathcal{H}_0 : \; Y = W \qquad\qquad \mathcal{H}_1 : \; Y = A + W$$

where W is a $\mathcal{N}(0, \sigma^2)$ random variable corresponding to additive noise at the receiver.

(a) Determine $f_0(y)$, the pdf of Y under $\mathcal{H}_0$, and also $f_1(y)$, the pdf of Y under $\mathcal{H}_1$.

(b) Determine the MAP decision rule assuming the priors π_0 and π_1 are known. Express the rule in terms of Y in the simplest way possible.

(c) Assume that $\pi_0 = \pi_1$. Determine the average error probability, p_e. You can leave your answer in terms of the Q or the Φ functions.

8. [**18 points**] Suppose X and Y are *zero-mean unit-variance* jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

   (a) Obtain $Var(3X - 2Y)$.

   (b) Obtain $P\{(3X - 2Y)^2 \leq 28\}$ in terms of the Q or the Φ functions.

   (c) Obtain $\mathsf{E}[Y \mid X = 3]$.

9. **[12 points]** Observations $X_1, \ldots, X_T$ produced by a drone's altimeter are assumed to have the form $X_t = bt + W_t$ where b is an unknown constant representing the rate of ascent of the drone (if $b < 0$ it means the drone is descending) and $W_1, \ldots, W_T$ represent observation noise and are assumed to be independent, $N(0, 1)$ random variables.

   (a) Write down the joint pdf of $X_1, \ldots, X_T$.

   (b) Obtain the maximum likelihood estimator of b for a particular vector of observations $x_1, \ldots, x_T$.

10. **[18 points]** Suppose U and V are independent random variables such that U is uniformly distributed over $[0,1]$ and V is uniformly distributed over $[0,2]$. Let $S = U + V$.

   (a) Obtain the mean and variance of S.

   (b) Derive and carefully sketch the pdf of S.

   (c) Obtain $\widehat{E}[U|S]$, the minimum mean square error linear estimator of U given S.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose X and Y are jointly continuous-type random variables with finite variance.

TRUE	FALSE	
☐	☐	If the MMSE for estimating Y from X is $\mathrm{Var}(Y)$, then X and Y must be uncorrelated.
☐	☐	If X and Y are uncorrelated then the MMSE for estimating Y from X is $\mathrm{Var}(Y)$.
☐	☐	If X and Y are uncorrelated and jointly Gaussian, then the MMSE for estimating Y from X is $\mathrm{Var}(Y)$.

(b) Let $X_1, \ldots, X_m$ be independent random variables, each with the binomial distribution with parameters 10 and p, where $0 < p < 1$, and let $S_m = X_1 + \ldots + X_m$.

TRUE	FALSE	
☐	☐	S_m has a binomial distribution
☐	☐	$\lim_{m\to\infty} P\left\{\frac{S_m}{m} \geq 10p(1-p)\right\} = 1$

(c) Consider a binary hypothesis testing problem. Let the subscript ML denote the maximum likelihood rule, and subscript MAP denote the maximum a posteriori rule.

TRUE	FALSE	
☐	☐	It is possible that $p_{\text{miss},ML} < p_{\text{miss},MAP}$.
☐	☐	It is possible that $p_{\text{false alarm},ML} = p_{\text{false alarm},MAP}$.
☐	☐	If $\pi_0 > \pi_1$ it is possible that $p_{\text{miss},ML} < p_{\text{miss},MAP}$.

(d) Let X and Y be uncorrelated, jointly Gaussian random variables, with parameters μ_X, μ_Y, σ_X^2 and σ_Y^2.

TRUE	FALSE	
☐	☐	$f_{XY}(u,v) = f_X(u)f_Y(v)$ for all real u, v
☐	☐	$E[XY] = 0$.