ECE 313 - Section X
Final Exam
Fall 2014

Name: ______________________________

NetID: ______________________________

- **Be sure that your exam booklet has 10 pages.**
- **Write your name at the top of each page.**
- This is a **closed book** exam.
- You may consult your three 8.5" x 11" sheets of notes.
- No calculators, cell phones, PDAs, tablets, or laptop computers are allowed.
- **Please show all your work.** Answers without appropriate justification will receive **very little** or **no credit**.
- If you need extra space, use the back of the previous page.

Problem 1	__________________	(25 pts)
Problem 2	__________________	(16 pts)
Problem 3	__________________	(12 pts)
Problem 4	__________________	(14 pts)
Problem 5	__________________	(18 pts)
Problem 6	__________________	(20 pts)
TOTAL	__________________	(105 pts)

Problem 1 (25 pts) – For each of the following parts, provide a short answer in the space provided. **Show your work and a justification** for your answer to get partial credit.

Part A (7 pts): We conduct an experiment where a single trial, i.e., a sample, consists of both Homa throwing a fair die (with 6 faces) and Keywhan throwing 2 fair coins.

I. (2 pts) Provide at least one example outcome of a trial (a single sample). What is the size of the sample space for this experiment?
Sample Space = {(1, T, T), (1, T , H), (1, H, H), …, (6, T, T),(6, T, H) , (6, H, H) }
Sample Space Size = 6 (faces) * 4 (2 coins)= 24

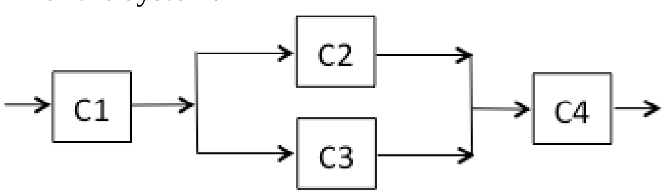
Let event A be "Homa getting an even number from a single throw of the die" and B be "Keywhan getting 2 heads from a throw of the 2 coins".

II. (2 pts) Test if events A and B are independent. Show your work.
Is independent
P(A) = ½ P(B) = ¼
P(A∩B) = 3/24 = 1/8 = P(A)P(B)
A∩B = {(2, H, H), (4, H, H), (6, H, H)}

III. (1 pts) Test if events A and B are mutually exclusive. Show your work.
P(A∩B) = 1/8 ≠ ϕ

IV. (2 pts) Define an event C such that A and C are mutually exclusive.
A: Homa gets an even number from a single throw of a dice = {2, 4, 6}
C: Homa gets an odd number from a single throw of a dice = {1, 3, 5}

Part B (6 pts): The following is the block diagram of a parallel-series system, composed of identical components whose time to failure is exponentially distributed with parameter λ. Calculate the reliability and hazard functions in terms of exponential distributions and determine the MTTF of the system.



Reliability Function: R(t) = R.(1- (1-R)^2).R = 2R^3 – R^4 = $2e^{-3\lambda t} - e^{-4\lambda t}$

Mean time to failure (MTTF): E[T]

$$= \int_0^\infty R(t)dt = \int_0^\infty 2e^{-3\lambda t} - e^{-4\lambda t}dt = \frac{2}{3\lambda} - \frac{1}{4\lambda} = \frac{5}{12\lambda}$$

Hazard Function: h(t) = $\frac{f(t)}{\boldsymbol{R}(t)}$ => $f(t) = -R'(t) = 6e^{-3\lambda t} - 4e^{-4\lambda t}$ => $h(t) = \frac{6e^{-3\lambda t} - 4e^{-4\lambda t}}{2e^{-3\lambda t} - e^{-4\lambda t}}$

Problem 1, continued:

Part C (6 pts) Suppose a total of 1000 Piazza posts are made during a semester, of which 60% are from students and the rest are from the instructors. Of the total posts, 30% were posted before the midterm. We define the following random variables:

$$W_k = \begin{cases} 1 & , if\ the\ k^{th}\ Piazza\ post\ is\ from\ a\ student \\ 0 & , otherwise \end{cases}$$

$$R_k = \begin{cases} 1 & , if\ the\ k^{th}\ Piazza\ post\ was\ made\ before\ the\ midterm \\ 0 & , otherwise \end{cases}$$

Assume W_k and R_k are independent random variables.

(a) Circle the distribution that best describes W_k and R_k:

| Binomial | Geometric | Bernoulli | Uniform |

(b) Var(W_k) = ?

$Var(W_k) = 0.6(1 - 0.6)$, 1pt

(c) Var($\sum_{k=0}^{1000} W_k$) = ?

$Var(\sum_{k=1}^{1000} W_k) = \sum_{k=1}^{1000} Var(W_k) = 1000 Var(W_k)$, 2pt

(d) What is the covariance of W_k and R_k (Cov(W_k, R_k))?

Without calculation

we know that the covariance is 0 for W and R being independent events., 2pt

Part D (6 pts) If X is a normally distributed random variable with parameters μ and σ, determine which of the following statements are TRUE and provide a short justification.

I. $Z = \frac{(X-\mu)}{\sigma}$ is normally distributed with mean 0 and variance of 1.

TRUE – $Standard\ normal$,1 pt

II. $F_Z(-z) = 1 - F_Z(\frac{X-\mu}{\sigma})$

TRUE – $F_Z(-z) = 1 - F_Z(z)$, 2pts

III. $W = aX + b$ is not normally distributed.

FALSE, 1 pt

IV. If Y_1, Y_2, …, Y_n is a sequence of independent, identically distributed random variables with mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$, then S = $Y_1 + Y_2 + \ldots + Y_n$ has the same distribution as X.

TRUE – $Central\ limit\ theorem \Rightarrow S\ normally\ distributed\ with\ \mu\ and\ \sigma$, 2 pts

Problem 2 (16 pts) – The memory failure data in a given month for a Super Computer is presented in the following table. The first column records the id of the memory module that was determined to be faulty, and the second column records the number of days since startup of the system.

Mem ID	**Days since startup**	**Mem ID**	**Days since startup**	**Mem ID**	**Days since startup**	**Mem ID**	**Days since startup**
A	1	F	10	K	22	P	26
B	3	G	15	L	23	Q	27
C	3	H	19	M	23	R	27
D	7	I	20	N	23	S	27
E	8	J	20	O	26		

Assuming that there were 100 memory modules functioning in the machine at the beginning of the month. Construct a table showing for each time interval, the failure density per hour and the hazard rate per hour for the data using the time interval of 7 days. (No need to simplify $f_d(t)$, $z_d(t)$)

Time Interval (days)	**Number of Failures in the interval**	**Failure Density $f_d(t)$ (/day)**	**Hazard Rate $h_d(t)$ or $z_d(t)$ (/day)**
0 – 7 ($0 \le$ days < 7)	3	3/(100*7)	3/(100*7)
7 – 14	3	3/(100*7)	3/(97*7)
14 - 21	4	4/(100*7)	4/(94*7)
21 – 28	9	9/(100*7)	9/(90*7)

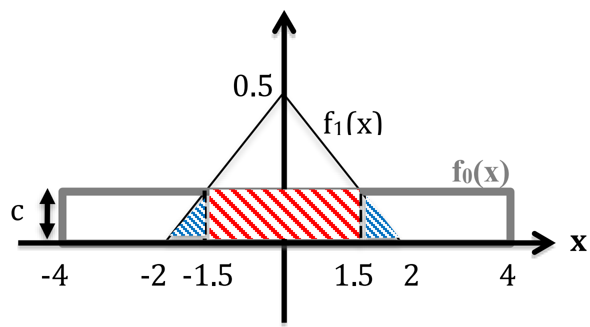
1pt each the rest
2pts each for hazard

Name: ____

Problem 3 (14 pts): As shown in the following figure, an observation X is drawn from a triangular distribution $f_1(x)$ with range [-2, 2] if hypothesis H_1 is true, and is drawn from a uniform distribution $f_0(x)$ with range [-4, 4] if hypothesis H_0 is true. Assume that the prior probability $P(H_0)$ is 0.8.

I. (3 pts) Find the value of c, where $f_0(x) = c$.

$\int_{-4}^{4} c = 1 \Rightarrow f_0(x) = \frac{1}{8}$



II. (5 pts) Use the figure above to find the ML decision rules and describe the rules in terms of X by filling in the following table.

III. (6 pts) Shade the *false alarm* and *miss detection* regions for the ML rule on the figure. Calculate the error probabilities for both ML rules by filling in the following table.

		ML Decision Rule	**MAP Decision Rule (Bonus)**
Part II	**H_0**	$1.5 < X$ or $X < -1.5$	For all X
	H_1	$-1.5 \le X$ and $X \le 1.5$	–
Part III	**Miss Detection Probability**	½ * 1 * 1/8	1
	False Alarm Probability	3 * 1/8	0
	Error Probability	0.2 * 1/16 + 0.8 * 3/8	0.2

Bonus (5 pts): Repeat parts II and III for MAP decision rule.

Problem 4 (12 pts) – Suppose an office has a single printer and a stack of 10 cartridges in the supply room. Let X be the number of pages that can be printed using a single toner with a mean of 1,000 pages.

I. (2 pts) Find the average number of pages that can be printed using the 10 cartridges.

$$10E[X] = 10{,}000$$

II. (3 pts) Estimate an upper bound for the probability that at least 30,000 pages can be printed.

$$P(X \geq 30{,}000) \leq \frac{E[X]}{30{,}000} = \frac{10{,}000}{30{,}000} = \frac{1}{3} = 0.33$$

III. (3 pts) Suppose that we also know that the variance of pages that can be printed using the 10 toners is 8,100. Estimate a lower bound for $P(|X - 10{,}000| < 100)$

K = 100

$P(-100 < X - 10{,}000 < 100) = 1 - P\{|X - E[X]| \geq 100)$

$P\{|X - E[X]| \geq 100) \leq 8100/10000 = 0.81$

$P\{|X - E[X]| < 100) > 1 - 0.81 = 0.19$

IV. (4 pts) If the numbers of pages that can be printed by cartridges are independent from each other, approximate the probability that the total number of pages that can be printed exceeds 10,180 using the central limit theorem.

$$p = P\left\{\frac{X_1 + \cdots + X_{10} - 10{,}000}{90} \geq \frac{10{,}180 - 10{,}000}{90}\right\} = P\{Z \geq 2\} = 1 - \phi(2)$$
$$= 1 - 0.9772 = 0.023$$

Problem 5 (18 pts) – Recall Problem 5 from midterm exam. The ECE 313 class is scheduled to start at 8:00am, but the instructor will not start the lecture until at least 3 students are present in the class. Now suppose that the start time of the class is also dependent on the functionality of the projector, i.e., the lecture cannot be started until the projector is working.

Suppose between 7:55am - 8:00am, the ECE 313 students arrive to the class with a rate of two students per minute. The student inter-arrival times are **exponentially distributed** and the student arrivals are independent of each other.

Suppose that from past experience, we know that on a given day, it takes on average 3 minutes for the projector to start (T1) and 5 minutes for the instructor and TAs to resolve any possible problems (T2). Let the time to successfully start the projector be represented by T. (**Hint:** If the projector fails T = T1 + T2, otherwise T = T1). Assume that T1 and T2 are exponentially distributed and independent from each other and that the projector fails with a probability of $p = 0.6$.

Part A (6 pts): Let N be the number of students that arrive between 7:55am - 8:00am. What is the probability of at least 3 students arriving to the class during this period $P(N_5 \geq 3)$?

Since the students inter-arrival times are exponentially distributed, the number of students arriving within the interval is a Poisson process with parameter λ = 2:

$$P(N_t = k) = e^{-2t}\frac{(2t)^k}{k!} \quad , k = 0,1,2,\ldots$$

$$P(N_5 \geq 3) = 1 - P(N_5 \leq 2) = 1 - e^{-2(5)}\left(\frac{(10)^0}{0!} + \frac{(10)^1}{1!} + \frac{(10)^2}{2!}\right) = 1 - 61e^{-10} = 0.997$$

Part B (6 pts): What distribution best describes the time to start the projector, if the projector fails (T = T1 + T2)? Write the pdfs of T1 and T2 and calculate the pdf of T.

T is sum of two independent sequential phases, each exponentially distributed with means of 3 minutes and 5 minutes. So the distribution of T is a two-stage hypo-exponential distribution $\lambda_1 = 1/3$ and $\lambda_2 = 1/5$ and its pdf would be written as follows:

$$f(t) = -\frac{1}{2}\left(e^{-\frac{t}{3}} - e^{-\frac{t}{5}}\right), t > 0$$

Part C (6 pts): What is the probability that the projector starts in less than 5 minutes, $P(T \leq 5)$? (**Hint:** Note that depending on the condition of the projector, the time to start the projector changes).

If the projector works, the time to start the project is T = T1 and exponentially distributed with $\lambda_1 = 1/3$, but if the projector fails to start, then the time to start it is T = T1+T2 and is a 2-stage hypo-exponential $\lambda_1 = 1/3$ and $\lambda_2 = 1/5$. We use the law of total probability to write the probability:

$$P(T \leq 5) = P(T \leq 5 \mid projector\ works)P(projector\ works) + P(T \leq 5 \mid projector\ fails)$$

$$P(T \leq 5) = 1 - P(T > 5) = 1 - [0.4 \int_5^{\infty} e^{-\frac{t}{3}} + 0.6 \int_5^{\infty} 2(e^{-\frac{t}{3}} - e^{-\frac{t}{5}})]$$

Part D – Bonus (3 pts): Suppose if the instructor arrives to class at 7:55am and starts the projector. What is the probability that the class starts on time?

The instructor will start the lecture if at least 4 students are present in the class and the projector is working. So for the class to be started on time, at least 4 students should arrive in the 5-minutes period of 7:55-8:00am ($N_5 \geq 4$) and the time to start the projector should be less than 5 ($T \leq 5$). Since these two events are independent from each other, the probability of the class starting on time can be written as follows:

$$P(Lecture\ on\ time) = P(T \leq 5).P(N_5 \geq 4)$$

Problem 6 (20 pts) – The random variables X and Y have the joint pdf:

$$f_{X,Y}(x,y) = ae^{-(x+y)} \quad for \quad \mathbf{0 < y < x < 1}$$

Part A (10 pts): Find the marginal pdfs of X and Y (First draw the region in which $f_{X,Y}(x,y)$ is defined).

$$f_X(x) = \int_0^x f_{X,Y}(x,y)dy = \int_0^x ae^{-(x+y)}dy = -ae^{-2x} + ae^{-x}$$

$$f_Y(y) = \int_y^1 f_{X,Y}(x,y)dy = \int_y^1 ae^{-(x+y)}dy = ae^{-2y} - ae^{-(1+y)}$$

Part B (6 pts): Find the probability of P(X + Y < 1) in terms of ***a*** (First draw the region of interest defined by X + Y < 1).

$$\int_0^{0.5} \int_y^{1-y} f_{X,Y}(x,y)dxdy = \int_0^{0.5} a(e^{-2y} - e^{-1})dy$$

$$= -\left[ae^{-1}y + \frac{a}{2}e^{-2y}\right]_0^{0.5}$$

$$= a(0.5 - e^{-1}) = 0.132a$$

Part C (4 pts): For what values of ***a*** are X and Y independent?

No values of a

$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ Hence they are not independent.

Table 2.3 Area $\Phi(x)$ under the Standard Normal Curve to the Left of x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5597	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8557	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998