University of Illinois Fall 2015

ECE 313: Final Exam

Monday, December 14, 2015
7 p.m. — 10 p.m.
Room LMS 141 , last name Aa-Lf
Room NOYES 100, last name Lg-Zz

1. [**12 points**] The three parts of this problem are unrelated.

   (a) Count the number of ways in which you can arrange the 8 letters in the word "computer".

   Solution: The string "computer" contains 8 characters. Therefore, the number of ways in which one can arrange the letters is 8!.

   (b) Count the number of distinct 10 letter sequences in which you can arrange the letters in the word "possession".

   Solution: The string "possession" contains 4 s's, 1 e's, 2 o's, 1 n, 1 p, and 1 i. Therefore, the number of distinct 10 letter sequences is $\frac{10!}{4!1!2!1!1!1!}$.

   (c) There are 6 balls in an urn: 2 blue, 2 green, and 2 red. Suppose there are 3 people present, and one at a time, they each randomly draw two balls out of the urn, without replacement.
   i) Find the probability that each person draws a matching pair of balls.
   ii) Find the probability that no person draws a matching pair of balls.

   Solution:
   i) The first person draws a matching pair of balls with probability 1/5 (the first ball can be anything while the second has to match the first one). Given that the first person drew a matching pair of balls, the second person draws a matching pair of balls with probability 1/3 (again, the first ball can be anything while the second has to match the first one). Given that the first and second persons drew matching pairs of balls, the third person draws a matching pair of balls with probability 1. Therefore, the probability that each person draws a matching pair of balls is $(1/5) \times (1/3) = 1/15$.
   ii) The first person draws non-matching balls with probability 4/5 (the first ball can be anything (say blue) while the second one can be anything but blue). At this point, we are left with 2 balls of the same color and 2 other balls of different colors. For the sake of simplicity, assume that we are left with 2 reds, 1 green, and 1 blue. Now, the second person can draw $\{R, G\}$, $\{G, R\}$, $\{R, B\}$, or $\{B, R\}$ each with probability 1/6. Notice that $\{G, B\}$ and $\{B, G\}$ are not allowed because the second person has to leave a non-matching pair of balls for the third person. Thus, the second person draws non-matching balls with probability $4/6 = 2/3$. Given this, the third person draws non-matching balls with probability 1. Therefore, the probability that no person draws a matching pair of balls is $(4/5) \times (2/3) = 8/15$.

2. [**16 points**] The mean, μ, of a Gaussian random variable X with variance $\sigma^2 = 1$ is to be estimated.

(a) It is observed that $X = u$. Write the likelihood of this observation as simply as possible in terms of μ and u.

Solution: Since $X \sim N(1, \sigma^2)$, the likelihood of $X = u$ is given by $f_X(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2}}$.

(b) Find $\hat{\mu}_{ML}$, the maximum likelihood estimate of μ, if it is observed that $X = u$. Simplify your answer as much as possible.

Solution: The maximum likelihood estimate can be obtained by solving for μ_{ML} such that $\frac{df_X(u)}{d\mu}\Big|_{\hat{\mu}_{ML}} = 0$. Observe that

$$\frac{df_X(u)}{d\mu} = \frac{1}{\sqrt{2\pi}}(u-\mu)e^{-\frac{(u-\mu)^2}{2}}.$$

One can see that $\frac{df_X(u)}{d\mu} > 0$ if $u > \mu$ and $\frac{df_X(u)}{d\mu} < 0$ if $u < \mu$. Therefore, $\hat{\mu}_{ML} = u$.

(c) Suppose now that you can observe the independent Gaussian random variables X_1, ..., X_n, all with the same mean, μ, and the same variance $\sigma^2 = 1$. It is observed that $(X_1, \cdots, X_n) = (x_1, \cdots, x_n)$ for some particular vector of real numbers, $(x_1, \cdots, x_n)$.

i) Write the likelihood of this observation and simplify as much as possible in terms of μ, and $(x_1, \cdots, x_n)$.

ii) Find $\hat{\mu}_{ML}$, the maximum likelihood estimate of μ, if it is observed that $(X_1, \cdots, X_n) = (x_1, x_2, \cdots, x_n)$. Simplify your answer as much as possible.

Solution:

i) Since $X_1, \cdots, X_n$ are independent, the likelihood of $(X_1, \cdots, X_n) = (x_1, x_2, \cdots, x_n)$ is the product of the individual likelihoods. Precisely,

$$\begin{aligned} f_{X_1,\cdots,X_n}(x_1, \cdots, x_n) &= f_{X_1}(x_1) \cdots f_{X_n}(x_n) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-\mu)^2}{2}} \cdots \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_n-\mu)^2}{2}} \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n} e^{-\frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2}. \end{aligned}$$

Taking the derivative of the likelihood given in part (c) and setting it to zero, we get that

$$\frac{df_{X_1,\cdots,X_n}}{d\mu} = \frac{1}{\left(\sqrt{2\pi}\right)^n} \left(\sum_{i=1}^n (x_i - \mu)\right) e^{-\frac{1}{2}\sum_{i=1}^n (x_i-\mu)^2} = 0.$$

This is achieved when $\sum_{i=1}^n (x_i - \mu) = \sum_{i=1}^n x_i - n\mu = 0$. One can see that $\frac{df_{X_1,\cdots,X_n}}{d\mu} > 0$ if $\frac{1}{n}\sum_{i=1}^n x_i > \mu$ and $\frac{df_{X_1,\cdots,X_n}}{d\mu} < 0$ if $\frac{1}{n}\sum_{i=1}^n x_i < \mu$. Therefore, $\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^n x_i$.

3. **[22 points]** Consider the experiment of rolling two fair dice. Define the two events $A = \{\text{both dice show the same number}\}$ and $B = \{\text{sum of numbers showing is} \leq 4\}$.

(a) Obtain $P(B|A)$ and determine if A and B are independent (explain).

Solution: Outcomes are equally likely, so $P(\text{event}) = \frac{|\text{event}|}{|\Omega|}$. Therefore, by the definition of conditional probability, $P(B|A) = \frac{P(AB)}{P(A)} = \frac{|AB|}{|A|} = \frac{2}{6} = \frac{1}{3}$. $P(B) = \frac{|B|}{|\Omega|} = \frac{6}{36} \neq P(B|A)$, hence they are not independent.

(b) Repeat the experiment 10 times and let X be the number of times that event A occurs. Obtain the pmf of X.

Solution: In each experiment trial, event A either occurs, or it doesn't occur, hence $X \sim Binomial(10, 1/6)$, because $P(A) = 1/6$. Therefore, $p_X(k) = \binom{10}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{10-k}$ for $k \in \{0, 1, \ldots, 10\}$.

(c) Let $Y = 2X$, with X defined as above. Obtain $E[Y]$, $Var(Y)$ and $E[Y^2]$.

Solution: By linearity of expectation $E[Y] = E[2X] = 2E[X] = 2(10)\frac{1}{6} = \frac{10}{3}$.
$Var(Y) = Var(2X) = 2^2 Var(X) = 4\left(10\frac{1}{6}\frac{5}{6}\right) = \frac{50}{9}$.
$E[Y^2] = Var(Y) + \mu_Y^2 = \frac{50}{9} + \left(\frac{10}{3}\right)^2 = \frac{50}{3}$.

(d) Are X and Y, as defined above, independent? Explain.

Solution: If indepedent, then $P\{X = j, Y = k\} = P\{X = j\}P\{Y = k\}$ for all j, k. However, $P\{X = 1, Y = 4\} = P\{X = 1, 2X = 4\} = 0 \neq P\{X = 1\}P\{Y = 4\}$.

4. **[18 points]** Let X and Y be random variables with $\sigma_Y^2 = \sigma_X^2 = 1$. It is known that the best linear estimator of Y given X is $\hat{E}[Y|X] = -X + 5$.

(a) Obtain the best unconstrained estimator of Y given X, $g^*(X)$; and obtain the resulting minimum mean-squared error (MMSE).

Solution: Recall that $\hat{E}[Y|X] = \rho_{X,Y}\sigma_Y\left(\frac{X-\mu_X}{\sigma_X}\right) + \mu_Y$. We can then deduce that $\rho_{X,Y} = -1$, because $\hat{E}[Y|X] = -X + 5$ and $\sigma_Y^2 = 1$. It is known that $\rho_{X,Y} = -1$ if and only if $Y = aX + b$ for some constants $a < 0$ and b. Therefore, there is no MSE using the best linear estimator, and hence the best unconstrained estimator of Y given X, $g^*(X)$ is equal to the best linear estimator, $g^*(X) = E[Y|X] = \hat{E}[Y|X] = -X + 5$.

(b) Obtain the best linear estimator of X given Y, $\hat{E}[X|Y]$.

Solution: From part (a), we know that $Y = aX + b = -X + 5$, hence $X = -Y + 5$ and hence $\hat{E}[X|Y] = -Y + 5$.

5. **[24 points]** Consider a random variable X with pdf:

$$f_X(x;\theta) = \begin{cases} (1+\theta)x^\theta, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

We want to decide H_1 or H_0 upon the observation of X given the following hypotheses:

$$\begin{aligned} H_0 &: \ \theta_0 = 1 \\ H_1 &: \ \theta_1 = 2 \end{aligned}$$

(a) Find the ML decision rule.
Solution: The likelihood ratio test for this problem is given as follows:

$$\Lambda(x) = \frac{f_X(x;\theta_1)}{f_X(x;\theta_0)} \geq \tau$$

for an appropriate threshold τ.

$$\Lambda(x) = \frac{(1+\theta_1)x^{\theta_1}}{(1+\theta_0)x^{\theta_0}} = \frac{3x^2}{2x} = \frac{3}{2}x \geq \tau.$$

Thus, the ML decision rule corresponds to deciding H_1 when

$$x \geq \frac{2}{3},$$

since $\tau = 1$ in this case.

(b) Find the MAP decision rule if the priors $\pi_0 = P(H_0)$ and $\pi_1 = P(H_1)$ satisfy $\pi_0 = 3\pi_1$.
Solution: As before, the MAP decision rule decides H_1 if

$$x \geq \frac{2}{3}3 = 2,$$

since $\tau = \frac{\pi_0}{\pi_1} = 3$ in this case. **Note:** Since $0 \leq x \leq 1$, the MAP decision rule always decides H_0!

(c) Compute $P_{\text{false alarm}}$ for each decision rule.
Solution: In the case of a single observation, the corresponding decision rules are: For the ML rule:

$$P_{false\ alarm} = P(H_1|H_0) = P\left\{X \geq \frac{2}{3}|H_0\right\} = \int_{2/3}^{1} 2x dx = \left[x^2\right]_{2/3}^{1} = 1 - \frac{4}{9} = \frac{5}{9}.$$

For the MAP rule:

$$P_{false\ alarm} = P(H_1|H_0) = 0.$$

(d) Compute $P_{\text{miss detection}}$ for each decision rule.
Solution: For the ML rule:

$$P_{miss\ detection} = P(H_0|H_1) = P\left\{X < \frac{2}{3}|H_1\right\} = \int_{0}^{2/3} 3x^2 dx = \left[x^3\right]_{0}^{2/3} = \frac{8}{27}.$$

For the MAP rule:

$$P_{miss\ detection} = P(H_0|H_1) = 1.$$

6. **[8 points]** Assume that a casino player has found a malfunctioning slot machine, which returns an average amount of \$40 every time the player plays. Additionally, the corresponding standard deviation of the amount the machine returns every time is \$5. In order to maximize his revenue, the player has constructed a number of fake slot machine coins so that he can play at no cost. If he plays for 100 times, with what probability will he win at least \$3900, assuming the trials are independent and identically distrbuted.? [Hint: Use Central Limit Theorem]

   Solution: Let X_i denote the amount that the slot machine returns the ith time the player plays. Then, using $n = 100$ fake coins, the player will win $S_n = X_1+X_2+\ldots+X_n$ dollars. Therefore,

   $$E[S_n] = 40n, \qquad \text{var}(S_n) = 25n.$$

   By employing the Central Limit Theorem,

   $$\frac{S_n - E[S_n]}{\sqrt{25n}} = \frac{S_n - 4000}{50}.$$

   is approximately distributed as $\mathcal{N}(0, 1)$. So

   $$P(S_n \geq 3900) = P\left(\frac{S_n - 4000}{50} \geq \frac{3900 - 4000}{50}\right) \approx Q(-2) = \Phi(2) \approx 0.9772.$$

7. **[28 points]** Emails arrive at Jim's inbox according to a Poisson process with rate $\lambda = 1$ per hour.

   (a) Find the probability that Jim receives at least 2 emails between noon and 2pm.
   Solution: N_2 is Poisson(2). So $P(N_2 \geq 2) = 1 - P(N_2 = 0) - P(N_2 = 1) = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$.

   (b) Let T_1 denote the arrival time of the first email. Find the pdf of T_1.
   Solution: T_1 is exponentially distributed with parameter $\lambda = 1$. So $f_{T_1}(t)$ is e^{-t} if $t > 0$ and 0 otherwise.

   (c) Find the mean and variance of T_1.
   Solution: $E[T_1] = \frac{1}{\lambda} = 1$, $\text{Var}(T_1) = \frac{1}{\lambda^2} = 1$.

   (d) Let T_2 denote the arrival time of the second email. Compute the correlation coefficient between T_1 and T_2.

   Solution: Let U denote the interarrival time between the first and second email. Then T_1 and U are iid. Then $\text{Var}(T_2) = \text{Var}(T_1) + \text{Var}(U) = 2\text{Var}(T_1)$ and $\text{Cov}(T_1, T_2) = \text{Cov}(T_1, T_1) = \text{Var}(T_1)$. Therefore $\rho_{T_1,T_2} = \frac{\text{Cov}(T_1,T_2)}{\sqrt{\text{Var}(T_1)\text{Var}(T_2)}} = \frac{1}{\sqrt{2}}$.

   (e) Suppose each email has a chance $\frac{1}{2}$ of being a spam email independently of everything else. Find the probability that Jim receives no spam from noon to 2pm.

Solution:

$$
\begin{aligned}
&P(\text{no spam from noon to 2pm}) \\
&= \sum_{k=0}^{\infty} P(\text{all } k \text{ emails are not spam}) P(\text{received } k \text{ emails from noon to 2pm}) \\
&= \sum_{k=0}^{\infty} 2^{-k} e^{-2} \frac{2^k}{k!} = e^{-1}.
\end{aligned}
$$

(f) Let S denote the arrival time of the first spam. Find its CDF and pdf.

Solution: By the same calculation as above, $1 - F_S(t) = P(S > t) = P(\text{no spam in } [0,t]) = \sum_{k=0}^{\infty} 2^{-k} e^{-t} \frac{(t)^k}{k!} = e^{-t/2}$. So S is exponentially distributed with parameter $1/2$, with CDF $F_S(t) = \begin{cases} 1 - e^{-t/2} & t > 0 \\ 0 & t \le 0 \end{cases}$ and pdf $f_S(t) = \begin{cases} \frac{1}{2} e^{-t/2} & t > 0 \\ 0 & t \le 0 \end{cases}$.

8. **[22 points]** Alice and Bob plan to go to lunch. They each arrive at a random time between noon and 1pm independently of each other. Denote the arrival time of Alice and Bob by X and Y respectively, which are both uniformly distributed over $[0,1]$ and mutually independent.

(a) Find the probability that Alice arrives earlier than Bob.

Solution: By symmetry, $P(X < Y) = 1/2$.

(b) Let Z denote the arrival time of the latest of the two. Find the pdf of Z and the mean $E[Z]$.

Solution: Let $Z = \max(X, Y)$. For any $z \in (0,1)$, $P(Z \le z) = P(X \le z, Y \le z) = P(X \le z) P(Y \le z) = z^2$. Therefore the pdf is $f_Z(z) = \begin{cases} 2z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$.

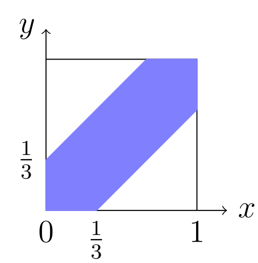
Then $E[Z] = \int_0^1 2z^2 = \frac{2}{3}$.

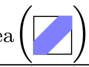
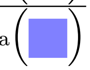
(c) Let W denote the arrival time of the earliest of the two. Find its mean $E[W]$.

Solution: Since $W = \min(X, Y)$ and $X + Y = Z + W$, $E[W] = E[X] + E[Y] - E[Z] = \frac{1}{3}$.

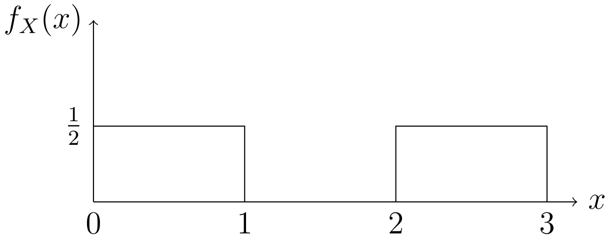
(d) Suppose both Alice and Bob are willing to wait for each other for at most 20 minutes, that is, whoever arrives the first will leave if the other person does not show up in 20 minutes. Find the probability that Alice and Bob meet.

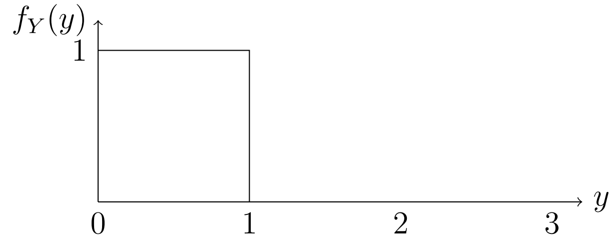
Solution: The region $\{(x,y) : 0 < x, y < 1, |x - y| \le \frac{1}{3}\}$ is drawn as follows.



Then $P(\text{Alice and Bob meet}) = P(|X - Y| \leq 1/3) =$ area() / area() $= \frac{5}{9}$.

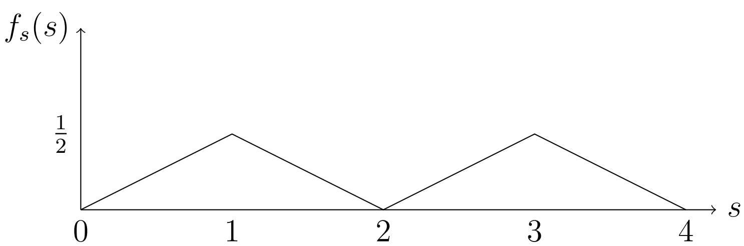
9. **[20 points]** Let X and Y be independent random variables with pdfs plotted below:



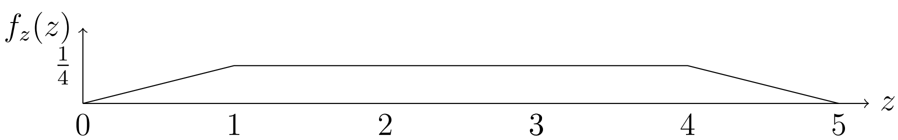


(a) Find the correlation $E[XY]$ and the correlation coefficient $\rho_{X,Y}$.
Solution: By independence $E[XY] = E[X]E[Y] = \frac{3}{4}$ and $\rho_{X,Y} = 0$.

(b) Let $S = X + Y$. Sketch the pdf f_S, clearly labelling all important points.
Solution:



(c) Let $Z = X + 2Y$. Sketch the pdf f_Z, clearly labelling all important points.
Solution:



10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose X and Y are random variables.

TRUE	FALSE	
☐	☐	If $Cov(X - aY, X + aY) = 0$, X and Y must be uncorrelated.
☐	☐	If $Cov(X, Y^2) = 1$, X and Y cannot be independent.
☐	☐	If $\rho_{X,Y} = 1$, there exists a real-valued constant a such that $Y = aX^2$.

Solution: False, True, False

(b) Let A, B be two events in the sample space Ω with non-zero probabilities. Let $E_1, E_2, \ldots, E_n$ be a partition of Ω.

TRUE	FALSE	
☐	☐	If A, B are mutually exclusive, then A, B must be independent.
☐	☐	$\sum_{i=1}^{n} P(E_i \vert A) = \sum_{i=1}^{n} P(E_i \vert B)$ always holds.
☐	☐	$\sum_{i=1}^{n} P(A \vert E_i) = \sum_{i=1}^{n} P(B \vert E_i)$ always holds.

Solution: False, True, False

(c) Assume that $X_1, X_2, \ldots, X_n$ are identically distributed random variables with mean $-\infty < \mu < +\infty$ and variance $0 < \sigma^2 < +\infty$. Let $S_n = X_1 + \ldots + X_n$.

TRUE	FALSE	
☐	☐	If $X_1, X_2, \ldots, X_n$ are also independent, then for any $\epsilon > 0$, $P\left(\left\vert\frac{S_n}{n} - \mu\right\vert \geq \epsilon\right) \to 0$ as $n \to +\infty$.
☐	☐	If $X_1 = X_2 = \cdots = X_n$, then for any $\epsilon > 0$, $P\left(\left\vert\frac{S_n}{n} - \mu\right\vert \geq \epsilon\right) \to 0$ as $n \to +\infty$.

Solution: True, False

(d) Let X and Y be random variables such that $P(X > 0) = 1$ and $P(Y < 0) = 1$.

TRUE	FALSE	
☐	☐	X and Y are always negatively correlated.
☐	☐	$E[XY]$ is always negative.

Solution: False, True