University of Illinois Fall 2015

ECE 313: Final Exam

Monday, December 14, 2015
7 p.m. — 10 p.m.
Room LMS 141 , last name Aa-Lf
Room NOYES 100, last name Lg-Zz

Name: (in BLOCK CAPITALS) ______________________________

NetID: ______________________________

Signature: ______________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 10 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 12 points ____________
2. 16 points ____________
3. 22 points ____________
4. 18 points ____________
5. 24 points ____________
6. 8 points ____________
7. 28 points ____________
8. 22 points ____________
9. 20 points ____________
10. 30 points ____________

Total (200 points) ____________

1. [**12 points**] The three parts of this problem are unrelated.

   (a) Count the number of ways in which you can arrange the 8 letters in the word "computer".

   (b) Count the number of distinct 10 letter sequences in which you can arrange the letters in the word "possession".

   (c) There are 6 balls in an urn: 2 blue, 2 green, and 2 red. Suppose there are 3 people present, and one at a time, they each randomly draw two balls out of the urn, without replacement.
   i) Find the probability that each person draws a matching pair of balls.
   ii) Find the probability that no person draws a matching pair of balls.

2. [**16 points**] The mean, μ, of a Gaussian random variable X with variance $\sigma^2 = 1$ is to be estimated.

   (a) It is observed that $X = u$. Write the likelihood of this observation as simply as possible in terms of μ and u.

   (b) Find $\hat{\mu}_{\text{M}L}$, the maximum likelihood estimate of μ, if it is observed that $X = u$. Simplify your answer as much as possible.

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(c) Suppose now that you can observe the independent Gaussian random variables X_1, ..., X_n, all with the same mean, μ, and the same variance $\sigma^2 = 1$. It is observed that $(X_1, \cdots, X_n) = (x_1, \cdots, x_n)$ for some particular vector of real numbers, $(x_1, \cdots, x_n)$.
i) Write the likelihood of this observation and simplify as much as possible in terms of μ, and $(x_1, \cdots, x_n)$.
ii) Find $\hat{\mu}_{\mathrm{ML}}$, the maximum likelihood estimate of μ, if it is observed that $(X_1, \cdots, X_n) = (x_1, x_2, \cdots, x_n)$. Simplify your answer as much as possible.

3. [**22 points**] Consider the experiment of rolling two fair dice. Define the two events $A = \{\text{both dice show the same number}\}$ and $B = \{\text{sum of numbers showing is} \leq 4\}$.

(a) Obtain $P(B|A)$ and determine if A and B are independent (explain).

(b) Repeat the experiment 10 times and let X be the number of times that event A occurs. Obtain the pmf of X.

(c) Let $Y = 2X$, with X defined as above. Obtain $E[Y]$, $Var(Y)$ and $E[Y^2]$.

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(d) Are X and Y, as defined above, independent? Explain.

4. **[18 points]** Let X and Y be random variables with $\sigma_Y^2 = \sigma_X^2 = 1$. It is known that the best linear estimator of Y given X is $\hat{E}[Y|X] = -X + 5$.

(a) Obtain the best unconstrained estimator of Y given X, $g^*(X)$; and obtain the resulting minimum mean-squared error (MMSE).

(b) Obtain the best linear estimator of X given Y, $\hat{E}[X|Y]$.

5. [**24 points**] Consider a random variable X with pdf:

$$f_X(x;\theta) = \begin{cases} (1+\theta)x^{\theta}, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

We want to decide H_1 or H_0 upon the observation of X given the following hypotheses:

$$\begin{aligned} H_0 &: \ \theta_0 = 1 \\ H_1 &: \ \theta_1 = 2 \end{aligned}$$

(a) Find the ML decision rule.

(b) Find the MAP decision rule if the priors $\pi_0 = P(H_0)$ and $\pi_1 = P(H_1)$ satisfy $\pi_0 = 3\pi_1$.

(c) Compute $P_{\text{false alarm}}$ for each decision rule.

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(d) Compute $P_{\text{miss detection}}$ for each decision rule.

6. **[8 points]** Assume that a casino player has found a malfunctioning slot machine, which returns an average amount of \$40 every time the player plays. Additionally, the corresponding standard deviation of the amount the machine returns every time is \$5. In order to maximize his revenue, the player has constructed a number of fake slot machine coins so that he can play at no cost. If he plays for 100 times, with what probability will he win at least \$3900, assuming the trials are independent and identically distrbuted.? [Hint: Use Central Limit Theorem]

7. **[28 points]** Emails arrive at Jim's inbox according to a Poisson process with rate $\lambda = 1$ per hour.

   (a) Find the probability that Jim receives at least 2 emails between noon and 2pm.

   (b) Let T_1 denote the arrival time of the first email. Find the pdf of T_1.

   (c) Find the mean and variance of T_1.

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(d) Let T_2 denote the arrival time of the second email. Compute the correlation coefficient between T_1 and T_2.

(e) Suppose each email has a chance $\frac{1}{2}$ of being a spam email independently of everything else. Find the probability that Jim receives no spam from noon to 2pm.

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(f) Let S denote the arrival time of the first spam. Find its CDF and pdf.

8. **[22 points]** Alice and Bob plan to go to lunch. They each arrive at a random time between noon and 1pm independently of each other. Denote the arrival time of Alice and Bob by X and Y respectively, which are both uniformly distributed over $[0, 1]$ and mutually independent.

   (a) Find the probability that Alice arrives earlier than Bob.

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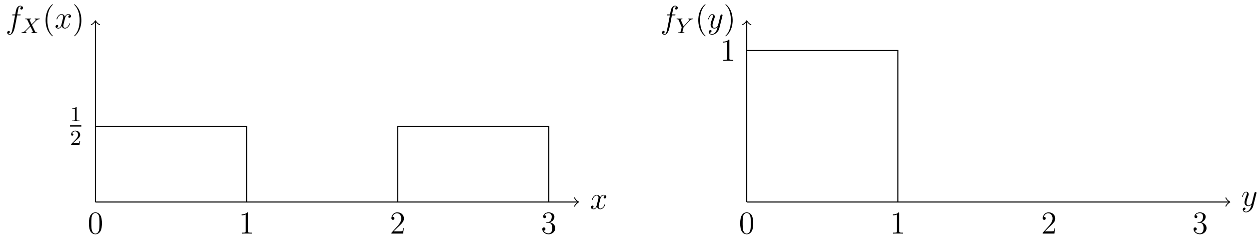
(b) Let Z denote the arrival time of the latest of the two. Find the pdf of Z and the mean $E[Z]$.

(c) Let W denote the arrival time of the earliest of the two. Find its mean $E[W]$.

(d) Suppose both Alice and Bob are willing to wait for each other for at most 20 minutes, that is, whoever arrives the first will leave if the other person does not show up in 20 minutes. Find the probability that Alice and Bob meet.

9. **[20 points]** Let X and Y be independent random variables with pdfs plotted below:



(a) Find the correlation $E[XY]$ and the correlation coefficient $\rho_{X,Y}$.

(b) Let $S = X + Y$. Sketch the pdf f_S, clearly labelling all important points.

(c) Let $Z = X + 2Y$. Sketch the pdf f_Z, clearly labelling all important points.

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose X and Y are random variables.

TRUE	FALSE	
☐	☐	If $Cov(X - aY, X + aY) = 0$, X and Y must be uncorrelated.
☐	☐	If $Cov(X, Y^2) = 1$, X and Y cannot be independent.
☐	☐	If $\rho_{X,Y} = 1$, there exists a real-valued constant a such that $Y = aX^2$.

(b) Let A, B be two events in the sample space Ω with non-zero probabilities. Let $E_1, E_2, \ldots, E_n$ be a partition of Ω.

TRUE	FALSE	
☐	☐	If A, B are mutually exclusive, then A, B must be independent.
☐	☐	$\sum_{i=1}^{n} P(E_i\|A) = \sum_{i=1}^{n} P(E_i\|B)$ always holds.
☐	☐	$\sum_{i=1}^{n} P(A\|E_i) = \sum_{i=1}^{n} P(B\|E_i)$ always holds.

(c) Assume that $X_1, X_2, \ldots, X_n$ are identically distributed random variables with mean $-\infty < \mu < +\infty$ and variance $0 < \sigma^2 < +\infty$. Let $S_n = X_1 + \ldots + X_n$.

TRUE	FALSE	
☐	☐	If $X_1, X_2, \ldots, X_n$ are also independent, then for any $\epsilon > 0$, $P\left(\left\|\frac{S_n}{n} - \mu\right\| \geq \epsilon\right) \to 0$ as $n \to +\infty$.
☐	☐	If $X_1 = X_2 = \cdots = X_n$, then for any $\epsilon > 0$, $P\left(\left\|\frac{S_n}{n} - \mu\right\| \geq \epsilon\right) \to 0$ as $n \to +\infty$.

(d) Let X and Y be random variables such that $P(X > 0) = 1$ and $P(Y < 0) = 1$.

TRUE	FALSE	
☐	☐	X and Y are always negatively correlated.
☐	☐	$E[XY]$ is always negative.