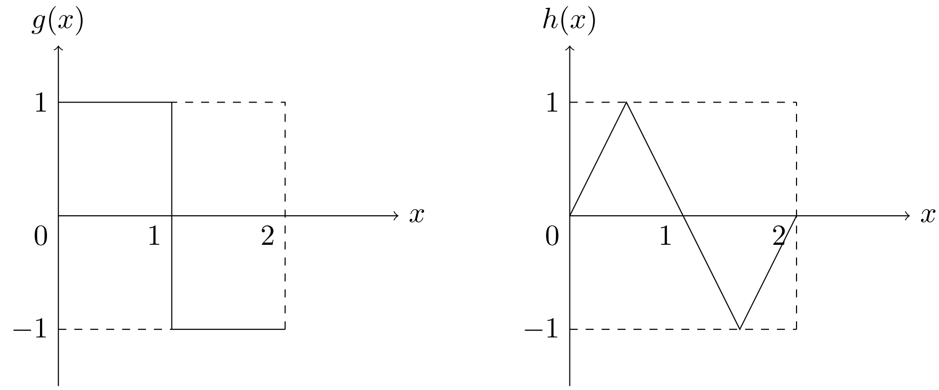
University of Illinois Fall 2015

ECE 313: Hour Exam II

Wednesday, November 11, 2015
7:00 p.m. — 8:15 p.m.

1. **[20 points]** Let X be uniformly distributed over the interval $[0, 2]$.



(a) (5 points) Let $Y = g(X)$, where $g(x)$ is the square wave plotted above. Is the distribution of Y of discrete-type or continuous-type? If discrete, find its probability mass function (pmf); if continuous, find its probability density function (pdf).

Solution: Discrete-type, since Y takes two values ± 1 with equal probability. So the pmf is $p_Y(1) = p_Y(-1) = 1/2$.

(b) (7 points) Let $Z = h(X)$, where $h(x)$ is the triangular wave plotted above. Is the distribution of Z of discrete-type or continuous-type? If discrete, find its pmf; if continuous, find its pdf.

Solution: Continuous-type and uniform on $[-1, 1]$. Consider $y \in (0, 1)$. Then $P(Y > y) = P(y/2 < X < 1 - y/2) = \frac{1-y}{2}$. So $f_Y(y) = \frac{1}{2}$ for $y > 0$ and by symmetry also for $y < 0$. So $f_Y(y) = \frac{1}{2}$ for $|y| < 1$ and zero otherwise, ie, Y is uniform over $(-1, 1)$.

(c) (4 points) Find the conditional probability $P(Y > 0 \mid Z > 0)$.

Solution: If $Z > 0$, then X lies in $[0, 1]$ and hence $Y = +1$ with probability one. So $P(Y > 0|Z > 0) = 1$.

(d) (4 points) Find a function r so that $r(X)$ is uniformly distributed over $[1, 5]$.

Solution: Set $r(X) = 2X + 1$.

2. **[22 points]** Let N_t be a Poisson process with rate $\lambda > 0$.

(a) (4 points) Obtain $P\{N_3 = 5\}$.

Solution: $N_3 \sim Poisson(3\lambda)$, so $P\{N_3 = 5\} = e^{-3\lambda}\frac{(3\lambda)^5}{5!}$.

(b) (6 points) Obtain $P\{N_7 - N_4 = 5\}$ and $E[N_7 - N_4]$.

Solution: $N_7 - N_4 \sim Poisson(3\lambda)$, so $P\{N_7 - N_4 = 5\} = e^{-3\lambda}\frac{(3\lambda)^5}{5!}$, and $E[N_7 - N_4] = 3\lambda$.

(c) (6 points) Obtain $P\{N_7 - N_4 = 5|N_6 - N_4 = 2\}$.

Solution: The increment $N_7 - N_6$ is independent of the increment $N_6 - N_4$, and $N_7 - N_6 \sim Poisson(\lambda)$, hence $P\{N_7 - N_4 = 5|N_6 - N_4 = 2\} = P\{N_7 - N_6 = 3\} = e^{-\lambda}\frac{(\lambda)^3}{3!}$.

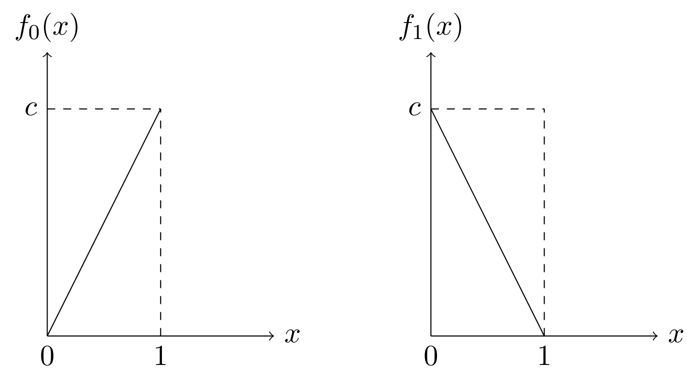
(d) (6 points) Obtain $P\{N_6 - N_4 = 2 | N_7 - N_4 = 5\}$.
Solution: Using the definition of conditional probability

$$P\{N_6 - N_4 = 2 | N_7 - N_4 = 5\} = \frac{P\{N_6 - N_4 = 2, N_7 - N_4 = 5\}}{P\{N_7 - N_4 = 5\}} = \frac{P\{N_6 - N_4 = 2, N_7 - N_6 = 3\}}{P\{N_7 - N_4 = 5\}}$$

$$= \frac{P\{N_6 - N_4 = 2\}P\{N_7 - N_6 = 3\}}{P\{N_7 - N_4 = 5\}} = \frac{\left(e^{-2\lambda}\frac{(2\lambda)^2}{2!}\right)\left(e^{-\lambda}\frac{(\lambda)^3}{3!}\right)}{e^{-3\lambda}\frac{(3\lambda)^5}{5!}} = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3,$$

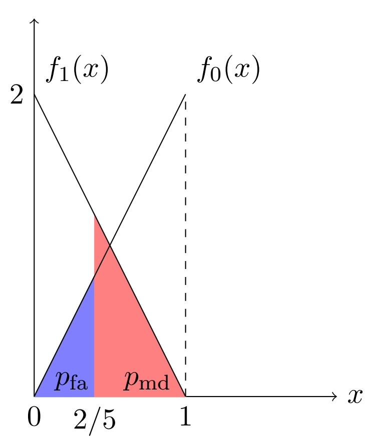
because $N_6 - N_4 \sim Poisson(2\lambda)$, $N_7 - N_6 \sim Poisson(\lambda)$, and $N_7 - N_4 \sim Poisson(3\lambda)$. This can also be done by realizing that, conditioned on 5 counts between $t = 4$ and $t = 7$, the arrival time of each one of those counts is uniformly distributed within the interval $(4, 7)$. The interval $(4, 6)$ is $\frac{2}{3}$ the length of the interval $(4, 7)$, hence we obtain $P\{N_6 - N_4 = 2 | N_7 - N_4 = 5\} = P\left\{Binomial\left(5, \frac{2}{3}\right) = 2\right\} = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3$.

3. **[20 points]** Let X be a continuous-type random variable taking values in $[0, 1]$. Under hypothesis H_0, the pdf of X is f_0; under hypothesis H_1, the pdf of X is f_1. Both pdfs are plotted below. The priors are known to be $\pi_0 = 0.6$ and $\pi_1 = 0.4$.



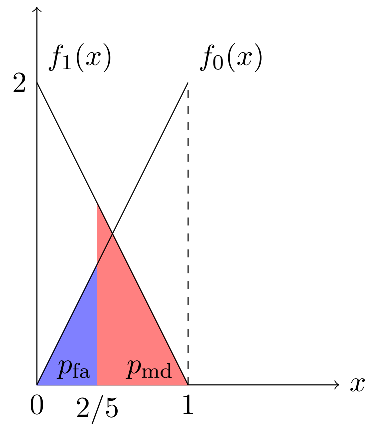
(a) (4 points) Find the value of c.
Solution: $c = 2$ to make sure the area under the density is one.

(b) (8 points) Specify the maximum a posteriori (MAP) decision rule for testing H_0 vs. H_1.
Solution:



The MAP rule declares H_1 if $f_1(x) > \frac{\pi_0}{\pi_1} f_0(x)$, i.e., $2 - 2x > 3x$. So declare H_1 if $x < 2/5$ and H_0 if $x > 2/5$.

(c) (8 points) Find the error probabilities $p_{\text{false alarm}}$, $p_{\text{miss detection}}$ and the average probability of error p_e for the MAP rule.
Solution:



The MAP rule declares H_1 if $f_1(x) > \frac{\pi_0}{\pi_1} f_0(x)$, i.e., $2 - 2x > 3x$. So declare H_1 if $x < 2/5$ and H_0 if $x > 2/5$. $p_{\text{false alarm}} = 4/25 = 0.16$, $p_{\text{miss detection}} = 9/25 = 0.36$, $p_e = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss detection}} = 6/25 = 0.24$.

4. **[20 points]** Let the joint pdf for the pair (X, Y) be

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le 1, 0 \le y \le 1, x + y \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

for some constant c.

(a) (5 points) Compute the marginal $f_X(x)$. You can leave it in terms of c.
Solution:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_0^{1-x} cxydy = \frac{c}{2}x(1-x)^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(b) (6 points) Obtain the value of the constant c for $f_{X,Y}$ to be a valid joint pdf.
Solution: Need $f_{X,Y}(x,y) \ge 0$ for all pairs (x,y), and

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = \int_0^1 \left\{ \int_0^{1-x} cxydy \right\} dx = \int_0^1 cx \left[\frac{y^2}{2} \right]_0^{1-x} dx =$$
$$\int_0^1 \frac{c}{2}x(1-x)^2 dx = \int_0^1 \frac{c}{2}(x-1+1)(x-1)^2 dx = \int_0^1 \frac{c}{2}(x-1)^3 dx + \int_0^1 \frac{c}{2}(x-1)^2 dx =$$
$$\frac{c}{2}\left[\frac{(x-1)^4}{4} \right]_0^1 + \frac{c}{2}\left[\frac{(x-1)^3}{3} \right]_0^1 = \frac{c}{2}\left(-\frac{1}{4} + \frac{1}{3} \right) = \frac{c}{24},$$

hence $c = 24$.

(c) (5 points) Obtain $P\left\{X + Y < \frac{1}{2}\right\}$.
Solution:

$$P\left\{X + Y < \frac{1}{2}\right\} = \int_0^{1/2} \int_0^{1/2-x} 24xydxdy = \frac{1}{16} = 0.0625.$$

(d) (4 points) Are X and Y independent? Explain why or why not.
Solution: The support of $f_{X,Y}$ is not a product set, hence they are not independent.

5. **[18 points]** Suppose we are testing if a transistor in a circuit is working properly or not. The voltage we observe at the output is a normal random variable X. If the transistor is working, X is distributed according to $N(1,1)$. If the transistor is not working, then X is distributed according to $N(-1,1)$. There is an 50% chance that the transistor is working. You can express the answers for this problem in terms of Φ and Q.

(a) (6 points) Find $P\{X \leq 1 | \text{transistor is working}\}$.
Solution: Let W be the event {transistor is working}. When the transistor is working, X follows a normal distribution with mean 1 and variance 1. Therefore,

$$P\{X \leq 1|W\} = \Phi\left(\frac{1-1}{1}\right) = \Phi(0) = 0.5.$$

(b) (6 points) Find $P\{X \geq 1\}$.
Solution: Then, using total probability and $P(W) = P(W^c) = \frac{1}{2}$,

$$\begin{aligned} P\{X \geq 1\} &= P\{X \geq 1|W\}P(W) + P\{X \geq 1|W^c\}P(W^c) \\ &= Q\left(\frac{1-1}{1}\right)\frac{1}{2} + Q\left(\frac{1-(-1)}{1}\right)\frac{1}{2} = \frac{1}{2}\frac{1}{2} + Q(2)\frac{1}{2} = \frac{1}{4} + \frac{1}{2}Q(2) \end{aligned}$$

(c) (6 points) Obtain the unconditional pdf of X, $f_X(u)$ for all u.
Solution: We first calculate the CDF of X for $-\infty < c < \infty$,

$$\begin{aligned} F_X(c) &= P(X \leq c) = P\{X \leq c|W\}P(W) + P\{X \leq c|W^c\}P(W^c) \\ &= \Phi\left(\frac{c-1}{1}\right)\frac{1}{2} + \Phi\left(\frac{c-(-1)}{1}\right)\frac{1}{2} = \Phi\left(c-1\right)\frac{1}{2} + \Phi\left(c+1\right)\frac{1}{2} \end{aligned}$$

Taking the derivative on both sides, we get that

$$f_X(c) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{(c-1)^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{(c+1)^2}{2}},$$

for $-\infty < u < \infty$.