1. **[14 points]** A tetrahedron has four faces, which are painted as follows: one side all red, one side all blue, one side all green, and one side with red, blue and green.

   ![Tetrahedron Diagram]

   Toss the tetrahedron randomly and the face that lands on the floor is equally likely among the four. Define the following events:

   \[ R = \{ \text{the face that hits the floor has red color} \} \]
   \[ B = \{ \text{the face that hits the floor has blue color} \} \]
   \[ G = \{ \text{the face that hits the floor has green color} \} \]

   (a) Compute the probabilities: \( P[R], P[G], P[B] \).

   **Solution:** \( P[R] = P[G] = P[B] = \frac{1}{2} \).

   (b) Are the events \( R, G, B \) pairwise independent? Justify your answer by calculations.

   **Solution:** \( P[RG] = P[GB] = P[BR] = P[4\text{th face}] = \frac{1}{4} \). So they are pairwise independent.

   (c) Are the events \( R, G, B \) independent? Justify your answer by calculations.

   **Solution:** \( P[RGB] = P[4\text{th face}] = \frac{1}{4} \neq P[R]P[G]P[B] \). So they are not independent.

2. **[15 points]** In a classroom there are \( n \) students whose birthday are equally likely chosen from \( k \) different dates.

   (a) One of the students is named Bob. Let \( X \) denote the number of other students who were born on the same day as Bob. What is the distribution of \( X \)?

   **Solution:** Binomial\((n-1, 1/k)\).

   (b) What is the probability that there is no other student who was born on the same day as Bob?

   **Solution:** \((1 - \frac{1}{k})^{n-1}\).

   (c) What is the probability that there is no pair of students in the classroom who were born on the same day? Consider the cases \( n \leq k \) and \( n > k \) separately.

   **Solution:** If \( n > k \), then there must exist a pair of students who are born on the same day so the probability is zero; if \( n \leq k \), then \( P[\text{no same birthday}] = \frac{k(k-1)\ldots(k-n+1)}{k^n} = (1 - \frac{1}{k})(1 - \frac{2}{k})\ldots(1 - \frac{n-1}{k}) \).

3. **[15 points]** The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson random variable with \( \lambda = 2 \).

   (a) What is the expected number of calls that will arrive during a 10-minute period?

   **Solution:** Let \( X \) be the number of phone calls arriving at a switchboard during any 10-minute period. \( X \) is Poisson distributed with parameter \( \lambda = 2 \). The expected number of calls in any 10-minute period is \( E[X] = \lambda = 2 \).
(b) Find the probability that more than three calls will arrive during a 10-minute period.

Solution: \( P(X > 3) = 1 - P(X \leq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) = 1 - e^{-2(1 + 2 + 2 + \frac{4}{3})} = 1 - \frac{19}{3}e^{-2}. \)

(c) Find the probability that no calls will arrive during a 10-minute period.

Solution: \( P(X = 0) = e^{-2}. \)

4. **[18 points]** Roll a fair die and let \( D \) be the number showing. Then, choose \( D \) cards at random (without replacement) from a standard deck of 52 cards and let \( K \) be the number of kings that you get.

(a) Obtain \( P\{K = 3|D = 4\} \).

Solution: Given that \( D = 4 \) there are \( \binom{4}{3} \) ways to choose 3 kings out of the 4 kings, and \( \binom{48}{1} \) ways to choose the remaining card out of the 48 that are not kings. Therefore,

\[
P\{K = 3|D = 4\} = \frac{\binom{4}{3} \binom{48}{1}}{\binom{52}{4}}.
\]

(b) Obtain \( P\{K = 3\} \).

Solution: Using total probability

\[
P\{K = 3\} = \sum_{i=1}^{6} P\{K = 3|D = i\}P\{D = i\} = \sum_{i=3}^{6} P\{K = 3|D = i\}P\{D = i\}
= \frac{4}{52} \left( \frac{1}{6} \right) + \frac{\binom{4}{3}}{\binom{52}{4}} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{1}}{\binom{52}{4}} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{2}}{\binom{52}{5}} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{3}}{\binom{52}{6}} \left( \frac{1}{6} \right)
\]

(c) Obtain \( P\{D = 4|K = 3\} \).

Solution: Using Bayes rule

\[
P\{D = 4|K = 3\} = \frac{P\{K = 3|D = 4\}P\{D = 4\}}{P\{K = 3\}} = \frac{\binom{4}{3} \binom{48}{1} \frac{1}{6}}{\frac{4}{52} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{1}}{\binom{52}{4}} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{2}}{\binom{52}{5}} \left( \frac{1}{6} \right) + \frac{\binom{4}{3} \binom{48}{3}}{\binom{52}{6}} \left( \frac{1}{6} \right)}
\]

5. **[18 points]** Consider the following \( s-t \) flow network, where link \( i \) has the indicated capacity \( C_i \), and link \( i \) fails with probability \( p_i \) independently of other links.

![Flow Network Diagram]

(a) What possible values of capacity can be achieved in this \( s-t \) flow network?

Solution: \( C \in \{0, 5, 10, 15\} \).
(b) Obtain the pmf of the capacity of this \( s - t \) flow network.

**Solution:** Let \( q_i = 1 - p_i \) and \( F_i = \{ \text{link fails} \} \) then

\[
p_C(0) = P\{ \text{all left links fail OR all right links fail} \} = P\{ F_1F_2F_3 \cup F_4F_5F_6 \} \\
= p_1p_2p_3 + p_4p_5p_6 - p_1p_2p_3p_4p_5p_6
\]

\[
p_C(5) = P\{ \text{only one right link works AND at least one left link works} \} \\
= P\{ (F_4F_5F_6^c \cup F_4F_5^cF_6 \cup F_4^cF_5F_6) (F_1F_2F_3) \} \\
= (p_4p_5q_6 + p_4q_5p_6 + q_4p_5p_6)(1 - p_1p_2p_3)
\]

\[
p_C(10) = P\{ \text{only two right links work AND at least one left links work} \} \\
= P\{ \{ (F_4F_5F_6^c \cup F_4^cF_5^cF_6 \cup F_4^cF_5F_6^c) (F_1F_2F_3) \} \cup \{ F_4^cF_5F_6^c (F_1F_2F_3^c \cup F_1F_2^cF_3 \cup F_1^cF_2F_3^c) \} \} \\
= (p_4q_5q_6 + q_4q_5p_6 + q_4p_5p_6)(1 - p_1p_2p_3) + q_4q_5q_6(p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3)
\]

\[
p_C(15) = P\{ \text{all right links work AND at least two left links work} \} \\
= P\{ (F_4^cF_5^cF_6^c (F_1F_2F_3^c \cup F_1^cF_2F_3^c \cup F_1^cF_2^cF_3)) \} \\
= q_4q_5q_6(p_1q_2q_3 + q_1q_2p_3 + q_1p_2q_3 + q_1q_2q_3)
\]

(c) Use the union bound to bound the probability of outage of this network.

**Solution:**

\[
p_C(0) = P\{ F_1F_2F_3 \cup F_4F_5F_6 \} \leq P\{ F_1F_2F_3 \} + P\{ F_4F_5F_6 \} = p_1p_2p_3 + p_4p_5p_6
\]

6. **[20 points]** Consider a random variable \( X \) uniformly distributed on the set \( \{1, \ldots, n\} \cup \{2n + 1, \ldots, 3n\} \), i.e. \( P\{ X = k \} \) is constant for \( k = 1, \ldots, n, 2n + 1, \ldots, 3n \).

(a) Suppose that \( n \) is unknown but it is observed that \( X = 9 \). Obtain the maximum likelihood estimate of \( n \).

**Solution:** \( P_X(k) = \frac{1}{3n} \) whenever \( k = 1, \ldots, n, 2n + 1, \ldots, 3n \), \( P_X(k) = 0 \) otherwise. Therefore, we would like to find the smallest \( n \) that is consistent with our observation \( X = 9 \). This is achieved when \( n = 3 \).

(b) Suppose now that it is known that \( n \) can have two different known values, \( n_1 \) and \( n_2 \), which gives rise to two hypotheses

\[
H_0 : X \in \{1, \ldots, n_1\} \cup \{2n_1 + 1, \ldots, 3n_1\}, \\
H_1 : X \in \{1, \ldots, n_2\} \cup \{2n_2 + 1, \ldots, 3n_2\},
\]

where \( n_1 < n_2 < 2n_1 \). Obtain the maximum likelihood decision rule.

**Solution:** \( P(X | H_0) = \frac{1}{3n_1} \) whenever \( X \in \{1, \ldots, n_1\} \cup \{2n_1 + 1, \ldots, 3n_1\} \), \( P(X | H_0) = 0 \) otherwise. Similarly, \( P(X | H_1) = \frac{1}{3n_2} \) whenever \( X \in \{1, \ldots, n_2\} \cup \{2n_2 + 1, \ldots, 3n_2\} \), \( P(X | H_1) = 0 \) otherwise. Since \( n_1 < n_2 \), \( P(X | H_0) > P(X | H_1) \) whenever \( X \in \{1, \ldots, n_1\} \cup \{2n_1 + 1, \ldots, 3n_1\} \). Therefore, the maximum likelihood rule will choose \( H_0 \) whenever \( X \in \{1, \ldots, n_1\} \cup \{2n_1 + 1, \ldots, 3n_1\} \), \( H_1 \) otherwise.

(c) Using the decision rule and distributions from part (b), obtain \( p_{\text{false alarm}} \).

**Solution:** \( p_{\text{false alarm}} = P(\text{say } H_1 | H_0 \text{ is true}) = 0 \) because the maximum likelihood decision rule will always say \( H_0 \) when \( H_0 \) is true.