University of Illinois Fall 2015

ECE 313: Hour Exam I

Wednesday, October 7, 2015
7:00 p.m. — 8:15 p.m.
A-J in MSEB 100, K-R in DCL 1320, S-Z in EVRT 151

Name: (in BLOCK CAPITALS) ________________________________

NetID: ________________________________

Signature: ________________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of 6 problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

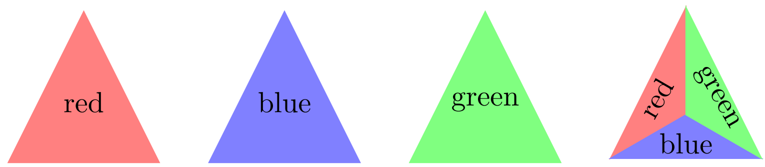
SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 14 points ____________
2. 15 points ____________
3. 15 points ____________
4. 18 points ____________
5. 18 points ____________
6. 20 points ____________

Total (100 points) ____________

1. **[14 points]** A tetrahedron has four faces, which are painted as follows: one side all red, one side all blue, one side all green, and one side with red, blue and green.



Toss the tetrahedron randomly and the face that lands on the floor is equally likely among the four. Define the following events:

$$R = \{\text{the face that hits the floor has red color}\}$$
$$B = \{\text{the face that hits the floor has blue color}\}$$
$$G = \{\text{the face that hits the floor has green color}\}$$

(a) Compute the probabilities: $P[R], P[G], P[B]$.

(b) Are the events R, G, B pairwise independent? Justify your answer by calculations.

(c) Are the events R, G, B independent? Justify your answer by calculations.

2. [**15 points**] In a classroom there are n students whose birthday are equally likely chosen from k different dates.

   (a) One of the students is named Bob. Let X denote the number of other students who were born on the same day as Bob. What is the distribution of X?

   (b) What is the probability that there is no other student who was born on the same day as Bob?

   (c) What is the probability that there is no pair of students in the classroom who were born on the same day? Consider the cases $n \leq k$ and $n > k$ separately.

3. **[15 points]** The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson random variable with $\lambda = 2$.

   (a) What is the expected number of calls that will arrive during a 10-minute period?

   (b) Find the probability that more than three calls will arrive during a 10-minute period.

   (c) Find the probability that no calls will arrive during a 10-minute period.

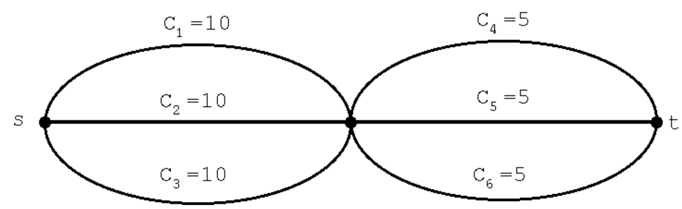
4. **[18 points]** Roll a fair die and let D be the number showing. Then, choose D cards at random (without replacement) from a standard deck of 52 cards and let K be the number of kings that you get.

   (a) Obtain $P\{K = 3|D = 4\}$.

   (b) Obtain $P\{K = 3\}$.

   (c) Obtain $P\{D = 4|K = 3\}$.

5. **[18 points]** Consider the following $s-t$ flow network, where link i has the indicated capacity C_i, and link i fails with probability p_i independently of other links.



(a) What possibles values of capacity can be achieved in this $s-t$ flow network?

(b) Obtain the pmf of the capacity of this $s-t$ flow network.

(c) Use the union bound to bound the probability of outage of this network.

6. **[20 points]** Consider a random variable X uniformly distributed on the set $\{1, \ldots, n\} \cup \{2n + 1, \ldots, 3n\}$, i.e. $P\{X = k\}$ is constant for $k = 1, \ldots, n, 2n + 1, \ldots, 3n$.

   (a) Suppose that n is unknown but it is observed that $X = 9$. Obtain the maximum likelihood estimate of n.

   (b) Suppose now that it is known that n can have two different known values, n_1 and n_2, which gives rise to two hypotheses

   $$H_0 : X \in \{1, ..., n_1\} \cup \{2n_1 + 1, ...3n_1\},$$

   $$H_1 : X \in \{1, ..., n_2\} \cup \{2n_2 + 1, ..., 3n_2\},$$

   where $n_1 < n_2 < 2n_1$. Obtain the maximum likelihood decision rule.

   (c) Using the decision rule and distributions from part (b), obtain $p_{\text{false alarm}}$.