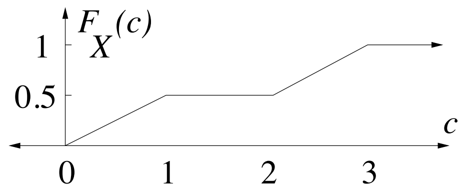
University of Illinois Fall 2014

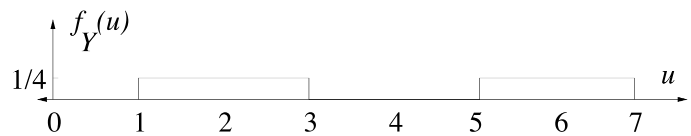
ECE 313: Hour Exam II

Wednesday, November 12, 2014
7:00 p.m. — 8:15 p.m.
Sect. B in 151 Everitt Lab, Sects. C&E in 100 Noyes Lab, Sect. D in 1404 Siebel Center

1. (a) Note that $f_X(u) = 0.5$ for $u \in [0,1] \cup [2,3]$ because the pdf must integrate to one.



(b) The pdf of Y has the same shape as f_X, but is translated and stretched, with height 1/4 over its support.



(c) Since f_X is symmetric about 1.5, E[X]=1.5. ALTERNATIVELY,

$$E[X] = \int_0^1 u\frac{1}{2}du + \int_2^3 u\frac{1}{2}du = \left.\frac{u^2}{4}\right|_0^1 + \left.\frac{u^2}{4}\right|_2^3 = \frac{1-0+9-4}{4} = 1.5$$

ALTERNATIVELY, we can use the area rule for expectations ($E[X]$ is the area bounded by the vertical axis, the horizontal line of height one, and the graph of F) to get $E[X] = 0.75 + 0.5 + 0.25 = 1.5$.

(d) Applying the function e^x to both sides of the inequality $\ln(X) \geq 1$ and using the fact $2 < e < 3$, shows that $P\{\ln(X) \geq 1\} = P\{X \geq e\} = \int_e^3 0.5du = \frac{3-e}{2}$.

2. (a) Let X denote the lifetime of an EBunny battery. We would like to know if

$$P\{X > 275\} > 0.99.$$

By switching to a standard Gaussian random variable $\widehat{X}$, we find

$$P\{X > 275\} = P\left\{\frac{X-300}{10} > \frac{275-300)}{10}\right\} = P\{\widehat{X} > -2.5\} = Q(-2.5) = \Phi(2.5) = 0.9938.$$

Since $0.9938 > 0.99$, the EBunny batteries meet the requirement.

(b) Since $E[Y] = np$ and $\mathrm{Var}(Y) = np(1-p)$, we solve the following two equations for n and p: $np = 300$ and $np(1-p) = 100$. Since $\frac{np(1-p)}{np} = 1-p$, we see $1 - p = \frac{100}{300}$, or $p = \frac{2}{3}$. Then $n = 300\left(\frac{3}{2}\right) = 450$ days.

3. (a) The solution follows by observing that $N(t+s) = (N(t+s) - N(t)) + N(t)$ and by using the independent increment property of Poisson processes:

$$E[N(t)N(t+s)] = E[N(t)(N(t+s) - N(t))] + E[N(t)^2] = \lambda t \lambda s + \lambda t + (\lambda t)^2.$$

(b) The event A of interest is

$$A = \{N(3) - N(1) = 3,\ N(3) - N(2) = 1,\ N(4) - N(2) = 1\}.$$

We can write A as the intersection of three independent events:

$$\begin{aligned} A &= \{N(2) - N(1) = 2,\ N(3) - N(2) = 1,\ N(4) - N(3) = 0\} \\ &= \{N(2) - N(1) = 2\} \cap \{N(3) - N(2) = 1\} \cap \{N(4) - N(3) = 0\} \end{aligned}$$

yielding $P(A) = \frac{e^{-2}2^2}{2!}\frac{e^{-2}2^1}{1!}\frac{e^{-2}2^0}{0!} = \frac{8e^{-6}}{2} = 4e^{-6}$.

4. (a) Since $f_X(u) = \frac{1}{2}$ for $u \in [-1, 1]$, LOTUS yields: $E[Y] = \int_{-1}^{1} u^4 \frac{1}{2} du = \frac{1}{5}$.

(b) We can start with the CDF of Y: for $v \leq 0$, it must be that $F_Y(v) = 0$. Analogously, for $v \geq 1$, it must be that $F_Y(v) = 1$. For $v \in (0, 1)$, we have $F_Y(v) = P\{Y \leq v\} = P\{-v^{\frac{1}{4}} \leq X \leq v^{\frac{1}{4}}\} = \frac{2v^{\frac{1}{4}}}{2} = v^{\frac{1}{4}}$. Finally, the pdf of Y is the derivative:

$$f_Y(v) = \begin{cases} \frac{1}{4}v^{-\frac{3}{4}} & 0 \leq v \leq 1 \\ 0 & \text{else} \end{cases}.$$

5. (a) The likelihood $f_X(\sqrt{2})$, or $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\sqrt{2}-\mu)^2}{2}}$, is maximized with respect to μ when $(\sqrt{2} - \mu)^2 = 0$, or the mean μ is equal to the observed value: so $\widehat{\mu}_{ML} = \sqrt{2}$.

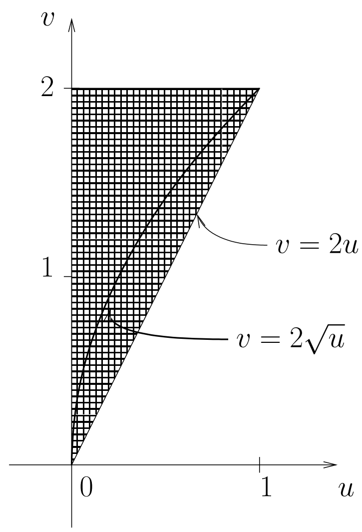
(b) Since $\frac{(\sqrt{2}-0)^2}{2\sigma^2} = \frac{1}{\sigma^2}$, the likelihood function $f_X(\sqrt{2})$ becomes $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{\sigma^2}}$. We need to find the value of σ^2 that maximizes it. The derivative of the logarithm of the likelihood is given by

$$\begin{aligned} \frac{d}{d\sigma^2}\left(constant - \frac{1}{2}\ln(\sigma^2) - \frac{1}{\sigma^2}\right) &= -\frac{1}{2\sigma^2} + \frac{1}{\sigma^4} \\ &= \frac{2 - \sigma^2}{2\sigma^4}, \end{aligned}$$

and it is positive if $\sigma^2 < 2$ and negative if $\sigma^2 > 2$. Thus, $\widehat{\sigma^2}_{ML} = 2$.

6. (a) No. For example, the support of f_{XY} is not a product set.

(b) For $0 \leq u \leq 1$, it holds that $2u \leq 2\sqrt{u} \leq 2$ (see figure below).



Therefore,

$$P\{Y > 2\sqrt{X}\} = \int_{u=0}^{1}\int_{v=2\sqrt{u}}^{2} 3u\,dv\,du = \int_0^1 6u(1-\sqrt{u})\,du = 6\left(\frac{1}{2}-\frac{2}{5}\right) = \frac{6}{10} = \frac{3}{5}.$$

(c)

$$f_X(u) = \begin{cases} \int_{v=2u}^{2} 3u\,dv = 6u(1-u) & \text{if } 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

(d) (Since $f_X(u)$ is non-zero only for $0 < u < 1$, the conditional pdf $f_{Y|X}(v|u)$ is well defined only for $0 < u < 1$.) For $0 < u < 1$,

$$f_{Y|X}(v|u) = \frac{f_{Y|X}(v|u)}{f_X(u)} = \begin{cases} \frac{3u}{6u(1-u)} = \frac{1}{2(1-u)} & \text{if } 2u \le v \le 2 \\ 0 & v < 2u \text{ or } v > 2 \end{cases}$$

i.e., conditioned on u, Y is uniformly distributed on the interval $[2u, 2]$.

Table 1: Φ function, the area under the standard normal pdf to the left of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990