University of Illinois **Fall 2013**

ECE 313: Problem Set 9
Functions of a Random Variable, Failure Rate Functions, Hypothesis Testing

Due: Wednesday, November 6 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.8-3.10, 4.1.

1. **[Function of a RV]**
   Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u, \; 0 < u < 1$, and 0 otherwise.
   (a) What is the average radius of the sphere?
   (b) What is the average volume?
   (c) What is the average surface area?
   (d) If a sphere of average radius is called an *average sphere*, then does an average sphere have more than, less than or equal the average volume? Does it have the average surface area?

2. **[Function of a RV]**
   $\mathbb{X}$ denotes a *uniform* random variable with mean 1 and variance 3. Find the pdf of $\mathbb{Y} = |\mathbb{X}|$.

3. **[Failure Rate Function]**
   Problem 3.36 of the course notes.

4. **[Hypothesis Testing]**
   Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathbb{X}$ is uniformly distributed on $(-1, 1)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is
   $$f_1(u) = \begin{cases} C(1 - |u|), & |u| < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Find the value of C.
   (b) Find the decision region Γ_0 for the *maximum-likelihood* decision rule.
   Remember that Γ_0 is the set of all real numbers such that if $\mathbb{X} \in \Gamma_0$, the decision is that H_0 is the true hypothesis.
   (c) Find the probabilities of false alarm and missed detection for the ML rule.
   (d) Suppose the prior probabilities of the two hypotheses H_0 and H_1 are π_0 and π_1, respectively. Find the decision rule Γ_0 for the MAP rule. Your answer should be in terms of the prior probabilities. *Hint*: First do this calculation for some numerical values of prior probabilities, say $\pi_0 = \frac{2}{3}, \pi_0 = \frac{3}{5}$, etc.
   (e) Calculate the probabilities of false alarm and missed detections for the MAP rule. Your answer should be in terms of the prior probabilities. *Hint*: First do this calculation for the same numerical values of prior probabilities you tried out in the previous part, say $\pi_0 = \frac{2}{3}, \pi_0 = \frac{3}{5}$, etc.
   (f) A student gets tired of all these decision rules and decides to use the "RAND" rule: toss a fair coin and decide H_0 if the outcome is heads and H_1 otherwise. What are the probabilities of the false alarm and missed detection with the RAND rule? For what values of π_0, if any, is the probability of false alarm with the RAND rule no larger than with the MAP rule?

5. **[Hypothesis Testing Concepts]**
   Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm

and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively. State whether the following statements are TRUE or FALSE. Include a short explanation.

(a) $P_{FA}^{ML} + P_{MD}^{ML} = 1$.

(b) $P_{FA}^{MAP} \leq P_{FA}^{ML}$.

(c) $P_{FA}^{ML}\pi_0 + P_{MD}^{ML}\pi_1 = 1$.

(d) $\pi_0 + \pi_1 = 1$.

(e) $P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 \geq P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$.

(f) If $P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 = P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$ then $\pi_0 = \pi_1$.

(g) If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.