University of Illinois **Fall 2013**

ECE 313: Problem Set 7
CDFs, Continuous Random Variables

Due: Wednesday, October 23 at 6 p.m.
Reading: *313 Course Notes* Sections 3.1-3.4

1. **[Using CDFs to find probabilities]**
   Problem 3.1 from the ECE 313 Course Notes (Section 3.12, page 145)

2. **[Validity of PDFs]**
   Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,

   - determine whether $f(u)$ is a valid probability density function (pdf).
   - If $f(u)$ is not a valid pdf, determine if there exists a constant C such that $C \cdot f(u)$ is a valid pdf.

   (a) $f(u) = 2u, \quad 0 < u < 1.$
   (b) $f(u) = |u|, \quad |u| < \frac{1}{2}$
   (c) $f(u) = 1 - |u|, \quad |u| < 1,$
   (d) $f(u) = \ln u, \quad 0 < u < 1.$ Hint: $\ln u$ can be integrated by parts.
   (e) $f(u) = \ln u, \quad 0 < u < 2,$
   (f) $f(u) = \frac{2}{3}(u-1), \quad 0 < u < 3,$
   (g) $f(u) = \exp(-2u), \quad u > 0.$
   (h) $f(u) = 4\exp(-2u) - \exp(-u), \quad u > 0,$
   (i) $f(u) = \exp(-|u|), \quad |u| < 1,$

3. **[Calculating probabilities from pdfs]**
   The continuous random variable $\mathbb{X}$ has pdf $f_{\mathbb{X}}(u) = \begin{cases} c(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

   (a) What is the value of c?
   (b) Find $P\{\mathbb{X} > 0.5\}$.
   (c) Find $P\{6\mathbb{X}^2 > 5\mathbb{X} - 1\}$.

4. **[Properties of uniformly distributed random variables]**
   Let $\mathbb{X}$ denote a uniformly distributed random variable. If $\mathsf{E}[\mathbb{X}] = 2$ and $\mathsf{var}(\mathbb{X}) = 3$, find $P\{\mathbb{X} \le 0\}$.

5. **[Using LOTUS]**
   Let $\mathbb{X}$ be a continuous random variable that is uniformly distributed on $[-1, +1]$.

   (a) Let $\mathbb{Y} = \mathbb{X}^2$. Calculate the mean and the variance of $\mathbb{Y}$.
   (b) Let $\mathbb{Z} = g(\mathbb{X})$ where $g(u) = \begin{cases} u^2, & u \ge 0, \\ -u^2, & u < 0. \end{cases}$ Find $\mathsf{E}[\mathbb{Z}]$.

6. **[Maximizing profits]**
   The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable $\mathbb{X}$ with probability density function

$$f_{\mathbb{X}}(u) = \begin{cases} 5(1-u)^4, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

   (a) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.

(b) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

(c) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathbb{Y}$ denote the amount of gasoline sold per week. How is $\mathbb{Y}$ related to $\mathbb{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit and how does it behave as a function of C, the tank capacity?

(d) Suppose that the owner pays \$$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly *net* profit and what value of C maximizes the average weekly net profit?

7. **[Exponential pdfs]**
The time $\mathbb{X}$ that a system that has been placed in operation time $t = 0$ breaks down is modeled as an exponential random variable $\mathbb{X}$ with parameter λ. $\mathbb{X}$ is called the *lifetime* of the system by optimists, while pessimists usually point out that $\mathbb{X}$ is actually the *time of death*. As the *Course Notes* tell you, $\mathsf{E}[\mathbb{X}] = \lambda^{-1}$ is the *mean lifetime* μ of the system.

(a) The *median lifetime* of a system (also called the *half-life* of a system) with lifetime $\mathbb{X}$ is the number m such that $F_{\mathbb{X}}(m) = \frac{1}{2}$. We expect that if we have a number of identical systems that were put into operation at time $t = 0$, then half of them will have failed at or before time m while half will still be operational.

What is the *median lifetime* m of the system with lifetime $\mathbb{X}$?

(b) The system costs \$$c_1$ to purchase and install, and \$$c_2$ per unit time to operate. If the system is still operating at time T, it is discarded and a brand-new replacement system installed. If the system fails before time T, nothing is done till time T when a brand-new replacement system is installed again. No *operating costs* are incurred from $\mathbb{X}$ till T, the time of re-installation, but there is a *penalty cost* of \$$c_3$ per unit time to have a non-operational system from $\mathbb{X}$ till T. Write an expression for $\mathbb{Y}$, the total cost of the system (purchase and installation cost + operational cost + penalty cost if any) during the time interval $[0, T)$ and find its expected value. As a special case, what is $\mathsf{E}[\mathbb{Y}]$ if T is chosen to be m, the median lifetime?

As a non-credit exercise, try and work out what choice of T minimizes the *average cost per unit time* $\mathsf{E}[Y/T]$ of this system.