University of Illinois **Fall 2013**

ECE 313: Problem Set 4
Geometric, negative binomial and Poisson distributions, Bernoulli process, ML parameter estimation

Due: Wednesday, September 25 at 6 p.m.
Reading: *313 Course Notes* Sections 2.5–2.8

1. [**Home Run Jack**]
Suppose that you are told that Home Run Jack hits a home run on average a quarter of the times he is at bat. Assume that the probability that Jack hits a home run in a time at bat is independent of any other time he is at bat.

   (a) Find the probability that it will take Jack more than 3 times at bat until he hits his next home run.

   (b) Find the mean and variance of the number of times at bat that you need to wait until Jack hits his next home run.

   (c) Suppose that Jack has been at bat 3 times already. Find the mean and variance of the number of times at bat that you still need to wait until Jack hits his next home run.

   (d) Suppose that you don't think that Jack really hits a home run a quarter of the times he is at bat. So, you start watching Jack at bat, and it takes him 5 times at bat until he hits a home run. What is the maximum likelihood estimate $\hat{p}_{ML}$ of the probability that Jack hits a home run at any given time at bat?

   (e) Suppose that you can't watch Jack at bat anymore, but after some time, you are told that he has hit 9 home runs out of his last 40 times at bat. Now, what is the maximum likelihood estimate $\hat{p}_{ML}$ of the probability that Jack hits a home run at any given time at bat?

   (f) Suppose instead that you are told that Jack got his 9th hit on his most recent (40th) turn at bat. Now, what is the maximum-likelihood estimate $\hat{p}_{ML}$ of the probability that Jack hits a home run at any given time at bat?

   (g) Suppose that someone else tells you that you were misinformed, and that Jack actually got his 9th hit on his 36th turn at bat. Now, what is the maximum-likelihood estimate $\hat{p}_{ML}$ of the probability that Jack hits a home run at any given time at bat?

   (h) Suppose that this last person didn't just tell you that Jack got his 9th hit on his 36th turn at bat, he also tells you that Jack did not hit a home run in his next four times at bat (turns 37 through 40). Now, what is the maximum likelihood estimate $\hat{p}_{ML}$ of the probability that Jack hits a home run at any given time at bat?

   (i) Comment on the similarities/differences between your answers to parts (e), (f), (g) and (h).

2. [**Maximum-likelihood Estimation**]
Let X denote a discrete random variable that takes on integer values $1, 2, \ldots, n$. The value of n is unknown, and we wish to find its maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that X had value 11 on a trial of the experiment.

   (a) Explain why $\hat{n}_{ML}$ must be 11 or more.

   (b) Suppose that X has the increasing-ramp pmf $p_X(k) = \begin{cases} \frac{2k}{n(n+1)}, & 1 \le k \le n, \\ 0, & otherwise. \end{cases}$

   What is $\hat{n}_{ML}$ in this case?

   (c) Suppose that X has the decreasing-ramp pmf $p_X(k) = \begin{cases} \frac{2(n+1-k)}{n(n+1)}, & 1 \le k \le n, \\ 0, & otherwise. \end{cases}$

Compute the value of $p_X(11)$ for $n = 11, 12, 13, \ldots$ to find the maximum-likelihood estimate $\hat{n}_{ML}$ numerically.

3. **[Rolling dice (again)]**
   Suppose you have a fair blue die and a fair red die, and define the following two random variables.

   (a) Let X denote the number of times one must roll the blue die until the outcome 4 has occurred 3 times. Find the pmf, the expected value and the variance of X.

   (b) Let Y denote the number of times one must roll the red die until the outcomes 1 or 2 have occurred four times. Find the pmf, the expected value and the variance of Y.

4. **[Customer support center]**
   Suppose the number of calls into a customer support center in any time interval is a Poisson random variable with mean 4 calls per minute.

   (a) What is the probability that there will be two calls in an interval of 3 minutes?

   (b) What is the probability that there is are at least three calls in an interval of one minute?

5. **[Overbooked flights]**
   Suppose that 105 passengers hold reservations for a 100-passenger flight from Chicago to Champaign, with each passenger showing up at the gate with probability 0.8, independently of any other passenger.

   (a) Find the probability that all the passengers that show up at the gate get a seat in the flight.

   (b) On average, how many passengers show up at the gate?

   (c) Explain why the number of *no-shows* can be modeled as a binomial random variable Y with parameters $(n = 105, p = 0.2)$.

   (d) Notice that the probability that everyone who shows up gets to go can also be expressed as $P\{Y \geq 5\}$. Use the *Poisson approximation* to compute $P\{Y \geq 5\}$ and compare your answer to the "more exact" answer that you found in part (a).