University of Illinois **Fall 2013**

ECE 313: Problem Set 3
Conditional probabilities, independence, and the binomial distribution

Due: Wednesday September 18 at 6 p.m.
Reading: *313 Course Notes* Sections 2.3–2.4

1. **[Scared chicken crossing a road]**
A chicken wants to cross the road but he is afraid he'll get run over. The chicken decides that it will toss a fair coin to decide whether or not to cross the road. If the coin toss is tails, it will not cross the road. The chicken is so scared that it decides that if the coin toss is heads, it will not cross the road immediately. Instead, it will toss the coin two more times. If the subsequent two tosses are both heads, then it will cross the road. If not, it will remain safely on the initial side of the road.
   (a) Find the probability that the chicken does not cross the road.
   (b) Given that the chicken did not cross the road, find the probability that the first toss is tails.
   (c) Given that the chicken did not cross the road, find the probability that the first toss is heads.

2. **[Elmo plays dice]**
Elmo was rummaging through his toy box and found three fair dice. He didn't like the fact that all six faces of each die were white, so he decided to paint some faces blue. He painted one face blue in one die, two faces blue on another die, and three faces blue in the remaining die. Elmo is happy now so he randomly chooses one of the dice (the three dice being equally likely to be chosen) and rolls it repeatedly.
   (a) Find the probability that the first roll shows a blue face.
   (b) Given that the first roll showed a blue face, find the conditional probability that the second roll also shows a blue face.
   (c) Given that the first roll showed a blue face, find the conditional probability that the second roll does not show a blue face.

3. **[Independence]**
This problem tests your understanding of independence. You may find part (a) to be useful in answering parts (b) or (c).
   (a) Prove the following statement: If A and B are independent and $B \subset A$, then either $P(B) = 0$ or $P(A) = 1$.
   (b) Consider tossing a fair coin 3 times. Are events $A =$"two or more tails" and $B =$"one or two heads" independent? Justify your answer.
   (c) Repeat part (b) with four coin tosses instead of three.

4. **[Snooping on your text messages]**
Suppose that Terry and Chris just started dating and that the two of them communicate through text messages. Terry's ex-boyfriend is jealous and has the ability to snoop on Terry's text messages without Terry knowing. Terry's ex is busy, so he can't snoop on all the text messages, he can only snoop on each one with probability 0.8, independently of the other text messages. Suppose that today, Terry and Chris sent each other a total of 8 text messages, and let X be the number of those eight text messages that were snooped on by Terry's ex.
   (a) Find the pmf of X and $E[X]$.
   (b) What is the most probable number of text messages snooped on by Terry's ex?
   (c) Given that at least one text message was snooped on, find the probability that exactly two text messages were snooped on.

(d) Suppose that if Terry's ex can snoop on at least 6 of the text messages, he can figure out Terry's plans for this weekend. Find the probability that he can figure out Terry's plans for this weekend.

5. **[Gambling with coin tosses]**
Suppose you are at your friendly neighborhood gambling joint and you are betting on coin tosses. In this game, a fair coin is tossed ten times and you win if at least four coin tosses show tails. Let X denote the number of tails observed in the 10 coin tosses, and let W denote the event that you win.

   (a) Find $P(W)$.

   (b) Find $P\{X \leq 5 \mid W\}$.

   (c) Given that $X = 4$, what is the probability that the 4th toss was a head?

   (d) You have a strong suspicion that the coin that was tossed is not a fair coin. Nonetheless, your friendly neighborhood bookmaker asks if you want to bet that the 4th toss was a head knowing only that the event $\{X = 4\}$ has occurred. The bookie knows the outcome of the 4th toss (but cannot change it after you have placed your bet!) as well as the value of $P(head) = p$. Does the fact that you don't know the value of p put you in a disadvantage? Why or why not?

6. **[Triple your money in five weeks?]**
A New Yorker runs an investment management service that has the stated goal of doubling the value of his clients' investments in a week via day trading. His brochure boasts that, "On average, my clients triple their money in five weeks!" After poring over back issues of the Wall Street Journal you learn the truth: at the end of any week, the investments of his clients will have doubled with probability 0.5, and will have decreased by 50% with probability 0.5. Thus, at the end of the first week, an initial investment of \$32 will be worth either \$64 or \$16, each with probability 0.5. Performance in any week is independent of performance during the other weeks. Anxious to apply your new skills in probability theory, you decide to invest \$32, and to let that investment ride for five weeks (in fact, you decide not to even look at the stock prices until the five weeks are over). Let the random variable X denote the value in dollars of your investment at the end of a five week period.

   (a) What are the possible values of X?

   (b) What is the pmf of the random variable X?

   (c) What is the expected value of X? Is the brochure accurate?

   (d) What is the probability that you will lose money on your investment?