1. [8 points] What is $E[X^2]$ for a Poisson random variable $X$ with mean 6?

Solution: $E[X^2] = \text{var}(X) + (E[X])^2$. For a Poisson random variable with parameter $\lambda$, $E[X] = \text{var}(X) = \lambda$ and thus $E[X^2] = 6 + 6^2 = 42$.

2. [8 points] If $Y$ is a geometric random variable with parameter $\frac{1}{2}$, what is $\text{var}(3 - 2Y)$?

Solution: $\text{var}(aY + b) = a^2 \text{var}(Y)$. Since $Y$ is given to be a geometric random variable, $\text{var}(Y) = \frac{1-p}{p^2} = \frac{1/2}{(1/2)^2} = 2$, we get that $\text{var}(3 - 2Y) = (-2)^2 \times 2 = 8$.

3. [8 points] Let $X$ denote the number of Heads that occur on 10 independent tosses of a coin with $P\{\text{Heads}\} = p$. Given that $m > 0$ heads were observed on the 10 tosses, what is the conditional probability that the second toss resulted in Heads?

Solution: Since $1 \leq m \leq 10$,

$$P\{\text{second toss Heads} \mid X = m\} = \frac{P\{\{\text{second toss Heads}\} \cap \{X = m\}\}}{P\{X = m\}}$$

$$= \frac{P\{\{\text{second toss Heads}\} \cap \{m - 1 \text{ Heads in 9 other tosses}\}\}}{P\{X = m\}}$$

$$= \frac{P\{\text{second toss Heads}\}P\{m - 1 \text{ Heads in 9 other tosses}\}}{P\{X = m\}}$$

$$= \frac{p \cdot \binom{9}{m-1}p^{m-1}(1-p)^{9-(m-1)}}{\binom{10}{m}p^m(1-p)^{10-m}} = \frac{9!}{(m-1)!(9-(m-1))!} \frac{10!}{m!(10-m)!} = \frac{m}{10^m}$$

4. [16 points] A biased coin is tossed repeatedly until a Head occurs for the first time. The probability of Heads in each toss is $\frac{1}{3}$. Let $X$ denote the number of tosses required until the first Head shows.

(a) [8 points] Find $E[X]$.

Solution: Notice that $X$ is a geometric random variable with parameter $p = 1/3$, hence $E[X] = 1/p = 3$.

Alternatively, notice that for integer $k > 0$, $P\{X = k\} = (1-p)^{k-1}p = q^{k-1}p$, where $q = 1-p$, so that

$$E[X] = \sum_{k=1}^{\infty} kP\{X = k\} = \sum_{k=1}^{\infty} kq^{k-1}p = p \frac{d}{dq} \left( \sum_{k=0}^{\infty} q^k \right) = p \frac{d}{dq} \left( \frac{1}{1-q} \right) = p \left( \frac{1}{(1-q)^2} \right)$$

$$= p \left( \frac{1}{p^2} \right) = \frac{1}{p} = 3.$$
Consider events appear than bug 2, and that the two bugs cannot exist simultaneously.

1 probability of two bugs. Bug 1 leads to error message 1 with probability \( P(A) = \frac{3}{5} \) and error message 2 with probability \( P(B) = \frac{2}{5} \). Bug 2 leads to error message 1 with probability \( \frac{1}{3} \) and error message 2 with probability \( \frac{2}{3} \). The student also has the knowledge that the bug 1 is 5 times more likely to appear than bug 2, and that the two bugs cannot exist simultaneously.

(a) [10 points] What is the probability that error message 1 appears?

Solution: Let \( E_1 \) denote the event of error message 1, \( B_1 \) denote the event of bug 1 and \( B_2 \) the event of bug 2.

\[
P(E_1) = P(E_1|B_1)P(B_1) + P(E_1|B_2)P(B_2) = \frac{1}{5} \times \frac{5}{6} + \frac{2}{3} \times \frac{1}{6} = \frac{5}{18}.
\]

(b) [10 points] Given that error message 1 appears, what is the probability that the system has bug 1?

Solution:

\[
P(B_1|E_1) = \frac{P(E_1|B_1)P(B_1)}{P(E_1)} = \frac{\frac{1}{5} \times \frac{5}{6}}{\frac{5}{18}} = \frac{3}{5}.
\]

6. [20 points] Consider events \( A, B, C, \) and \( D \) with probabilities \( P(A) = \frac{1}{5}, P(B) = \frac{3}{5}, P(C) = \frac{2}{5}, \) and \( P(D) = \frac{3}{5} \), and suppose that \( P(B \mid A) = \frac{1}{2} \).

(a) [7 points] Find \( P(A^cB) \) and \( P(A \mid B) \).

Solution:

\[
P(A^cB) = P(B) - P(AB) = P(B) - P(B \mid A)P(A) = \frac{3}{5} - \frac{1}{2} \times \frac{1}{5} = \frac{1}{2}.
\]

\[
P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{\frac{1}{2} \times \frac{1}{5}}{\frac{3}{5}} = \frac{1}{6}.
\]

(b) [5 points] If \( A \) and \( C \) are independent, find \( P(A^cC) \).

Solution:

\[
P(A^cC) = P(A^c)P(C) = \left(1 - \frac{1}{5}\right) \frac{2}{5} = \frac{8}{25}.
\]

(c) [8 points] If \( A \) and \( D \) are mutually exclusive, find as tight upper and lower bounds as possible on \( P(BD) \).

Solution: Since \( A \cap D = \emptyset \), and so \( ABD \subset AD \) is also the empty set, we get that

\[
BD = A^cBD \cup ABD = A^cBD \cup \emptyset = A^cBD \quad \Rightarrow \quad P(BD) = P(A^cBD).
\]

Now, since \( A^cBD \subset A^cB \) and we know from part (a) that \( P(A^cB) = \frac{1}{2} \), we have that

\[
P(BD) = P(A^cBD) \leq P(A^cB) = \frac{1}{2}.
\]
Now, \( \frac{3}{5} = P(D) = P(AD) + P(A^cD) = P(A^cD), \) and since \((A^cB \cup A^cD) \subset A^c\), we get that

\[
P(A^cB \cup A^cD) = P(A^cB) + P(A^cD) - P(A^cBD)
\]

\[
= \frac{1}{2} + \frac{3}{5} - P(BD)
\]

\[
= \frac{11}{10} - P(BD)
\]

\[
\leq P(A^c) = \frac{4}{5}
\]

giving us that \( P(BD) \leq \frac{3}{10} \). Thus,

\[
\frac{3}{10} \leq P(BD) \leq \frac{1}{2}.
\]

7. **[20 points]** Consider a regular 8x8 chessboard, which consists of 64 squares in 8 rows and 8 columns.

(a) **[10 points]** How many different rectangles, comprised entirely of chessboard squares, can be drawn on the chessboard? *Hint:* there are 9 horizontal and 9 vertical lines in the chessboard.

**Solution:** A rectangle is uniquely described by the pair of horizontal lines and the pair of vertical lines that form its sides. Since there are \( \binom{9}{2} = \frac{9 \times 8}{1 \times 2} = 36 \) choices for the pair of horizontal lines and, similarly, 36 choices for the pair of vertical lines, there are \( \binom{9}{2}^2 = 36 \times 36 = 1296 \) rectangles.

(b) **[10 points]** One of the rectangles you counted in part (a) is chosen at random. What is the probability that it is square shaped?

**Solution:** Number the horizontal and vertical lines as 0 through 8. Then, for any given \( k, 1 \leq k \leq 8 \), we get a \( k \times k \) square if we pick the pair of horizontal lines as one of \( (0, k), (1, k+1), (2, k+2), \ldots, (8-k, 8) \) and similarly for the pair of vertical lines. Thus, the number of \( k \times k \) squares is \( (9-k)^2 \). Hence, the number of square shaped rectangles is

\[
1^2 + 2^2 + 3^2 + \cdots + 8^2 = 12 \cdot 17.
\]

So the probability of a getting a square shaped rectangle is \( \frac{17}{36 \times 36} = \frac{17}{108} \).