University of Illinois Fall 2013

ECE 313: Hour Exam I

Monday October 14, 2013
7:00 p.m. — 8:20 p.m.
Library 66, Sections X and C (9 a.m. and 10 a.m.)
Altgeld Hall 314, Sections D and E (11 a.m. and 1 p.m.)

1. **[8 points]** What is $\mathsf{E}[X^2]$ for a Poisson random variable X with mean 6 ?

   Solution: $\mathsf{E}[X^2] = \mathsf{var}(X) + (\mathsf{E}[X])^2$. For a Poisson random variable with parameter λ, $\mathsf{E}[X] = \mathsf{var}(X) = \lambda$ and thus $\mathsf{E}[X^2] = 6 + 6^2 = 42$.

2. **[8 points]** If Y is a geometric random variable with parameter $\frac{1}{2}$, what is $\mathsf{var}(3 - 2Y)$?

   Solution: $\mathsf{var}(aY + b) = a^2\mathsf{var}(Y)$. Since Y is given to be a geometric random variable, $\mathsf{var}(Y) = \frac{1-p}{p^2} = \frac{1/2}{(1/2)^2} = 2$, we get that $\mathsf{var}(3 - 2Y) = (-2)^2 \times 2 = 8$.

3. **[8 points]** Let X denote the number of Heads that occur on 10 independent tosses of a coin with $P\{\text{Heads}\} = p$. Given that $m > 0$ heads were observed on the 10 tosses, what is the conditional probability that the second toss resulted in Heads?

   Solution: Since $1 \leq m \leq 10$,

$$\begin{aligned}
P\{\text{second toss Heads} \mid X = m\} &= \frac{P\{\{\text{second toss Heads}\} \cap \{X = m\}\}}{P\{X = m\}} \\
&= \frac{P\{\{\text{second toss Heads}\} \cap \{m - 1 \text{ Heads in 9 other tosses}\}\}}{P\{X = m\}} \\
&= \frac{P\{\text{second toss Heads}\}P\{m - 1 \text{ Heads in 9 other tosses}\}}{P\{X = m\}} \\
&= \frac{p \cdot \binom{9}{m-1} p^{m-1}(1 - p)^{9-(m-1)}}{\binom{10}{m} p^m (1 - p)^{10-m}} = \frac{\frac{9!}{(m-1)!(9-(m-1))!}}{\frac{10!}{m!(10-m)!}} = \frac{m}{10}.
\end{aligned}$$

4. **[16 points]** A biased coin is tossed repeatedly until a Head occurs for the first time. The probability of Heads in each toss is $\frac{1}{3}$. Let X denote the number of tosses required until the first Head shows.

   (a) **[8 points]** Find $E[X]$.
   Solution: Notice that X is a geometric random variable with parameter $p = 1/3$, hence $E[X] = 1/p = 3$.
   Alternatively, notice that for integer $k > 0$, $P\{X = k\} = (1 - p)^{k-1}p = q^{k-1}p$, where $q = 1 - p$, so that

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP\{X = k\} = \sum_{k=1}^{\infty} kq^{k-1}p = p\frac{d}{dq}\left(\sum_{k=0}^{\infty} q^k\right) = p\frac{d}{dq}\left(\frac{1}{1 - q}\right) = p\left(\frac{1}{(1 - q)^2}\right) \\
&= p\left(\frac{1}{p^2}\right) = \frac{1}{p} = 3.
\end{aligned}$$

(b) **[8 points]** Find $P\{X = 7 \mid X > 5\}$.
Solution: By the memoryless property of the geometric distribution,
$P\{X = 7 \mid X > 5\} = P\{X = 7 - 5\} = P\{X = 2\} = (1-p)p = \left(1 - \frac{1}{3}\right)\frac{1}{3} = \frac{2}{9}$.
To do this in more detail,
$P\{X = 7 \mid X > 5\} = \frac{P\{X=7, X>5\}}{P\{X>5\}} = \frac{P\{X=7\}}{P\{X>5\}} = \frac{(1-p)^6 p}{(1-p)^5} = (1-p)p = \left(1 - \frac{1}{3}\right)\frac{1}{3} = \frac{2}{9}$.

5. **[20 points]** While debugging a software program, a student has narrowed it down to one of two bugs. Bug 1 leads to error message 1 with probability $\frac{1}{5}$ and error message 2 with probability $\frac{4}{5}$. Bug 2 leads to error message 1 with probability $\frac{2}{3}$ and error message 2 with probability $\frac{1}{3}$. The student also has the knowledge that the bug 1 is 5 times more likely to appear than bug 2, and that the two bugs cannot exist simultaneously.

(a) **[10 points]** What is the probability that error message 1 appears?
Solution: Let $E1$ denote the event of error message 1, $B1$ denote the event of bug 1 and $B2$ the event of bug 2.

$$P(E1) = P(E1|B1)P(B1) + P(E1|B2)P(B2) = \frac{1}{5} \times \frac{5}{6} + \frac{2}{3} \times \frac{1}{6} = \frac{5}{18}.$$

(b) **[10 points]** Given that error message 1 appears, what is the probability that the system has bug 1?
Solution:

$$P(B1|E1) = \frac{P(E1|B1)P(B1)}{P(E1)} = \frac{\frac{1}{5} \times \frac{5}{6}}{\frac{5}{18}} = \frac{3}{5}.$$

6. **[20 points]** Consider events A, B, C, and D with probabilities $P(A) = 1/5$, $P(B) = 3/5$, $P(C) = 2/5$, and $P(D) = 3/5$, and suppose that $P(B \mid A) = 1/2$.

(a) **[7 points]** Find $P(A^c B)$ and $P(A \mid B)$.
Solution:

$$\begin{aligned} P(A^c B) &= P(B) - P(AB) = P(B) - P(B \mid A)P(A) = \frac{3}{5} - \frac{1}{2} \times \frac{1}{5} = \frac{1}{2} \\ P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B)} = \frac{\frac{1}{2} \times \frac{1}{5}}{\frac{3}{5}} = \frac{1}{6} \end{aligned}$$

(b) **[5 points]** If A and C are independent, find $P(A^c C)$.
Solution:

$$P(A^c C) = P(A^c)P(C) = \left(1 - \frac{1}{5}\right)\frac{2}{5} = \frac{8}{25}$$

(c) **[8 points]** If A and D are mutually exclusive, find as tight upper and lower bounds as possible on $P(BD)$.
Solution: Since $A \cap D = \emptyset$, and so $ABD \subset AD$ is also the empty set, we get that

$$BD = A^c BD \cup ABD = A^c BD \cup \emptyset = A^c BD \quad \Rightarrow \quad P(BD) = P(A^c BD).$$

Now, since $A^c BD \subset A^c B$ and we know from part (a) that $P(A^c B) = \frac{1}{2}$, we have that

$$P(BD) = P(A^c BD) \le P(A^c B) = \frac{1}{2}.$$

Now, $\frac{3}{5} = P(D) = P(AD) + P(A^cD) = P(A^cD)$, and since $(A^cB \cup A^cD) \subset A^c$, we get that

$$\begin{aligned} P(A^cB \cup A^cD) &= P(A^cB) + P(A^cD) - P(A^cBD) \\ &= \frac{1}{2} + \frac{3}{5} - P(BD) \\ &= \frac{11}{10} - P(BD) \\ &\leq P(A^c) = \frac{4}{5} \end{aligned}$$

giving us that $P(BD) \leq \frac{3}{10}$. Thus,

$$\frac{3}{10} \leq P(BD) \leq \frac{1}{2}.$$

7. **[20 points]** Consider a regular 8x8 chessboard, which consists of 64 squares in 8 rows and 8 columns.

   (a) **[10 points]** How many different rectangles, comprised entirely of chessboard squares, can be drawn on the chessboard? *Hint:* there are 9 horizontal and 9 vertical lines in the chessboard.

   Solution: A rectangle is uniquely described by the pair of horizontal lines and the pair of vertical lines that form its sides. Since there are $\binom{9}{2} = \frac{9\times8}{1\times2} = 36$ choices for the pair of horizontal lines and, similarly, 36 choices for the pair of vertical lines, there are $\binom{9}{2}^2 = 36 \times 36 = 1296$ rectangles.

   (b) **[10 points]** One of the rectangles you counted in part (a) is chosen at random. What is the probability that it is square shaped?

   Solution: Number the horizontal and vertical lines as 0 through 8. Then, for any given k, $1 \leq k \leq 8$, we get a $k \times k$ square if we pick the pair of horizontal lines as one of $(0, k), (1, k+1), (2, k+2), \ldots, (8-k, 8)$ and similarly for the pair of vertical lines. Thus, the number of $k \times k$ squares is $(9-k)^2$. Hence, the number of square shaped rectangles is $1^2 + 2^2 + 3^2 + \cdots + 8^2 = 12 \cdot 17$. So the probability of a getting a square shaped rectangle is $\frac{12\cdot17}{36\times36} = \frac{17}{108}$.