University of Illinois Fall 2013

ECE 313: Second MidTerm Exam

Monday November 18, 2013
7:00 p.m. — 8:20 p.m.

1. **[6 points]** The random variable $\mathbb{X}$ is uniformly distributed over the interval $[-1, 1]$. What is the value of $\mathsf{var}(\mathbb{X}^3)$?

2. **[6 points]** $\mathbb{X}$ is a Gaussian random variable (mean 60, variance 400) that models the average daily temperature (in °F) in a certain city.
   What is the probability that the average daily temperature is below 0°F?

3. **[6 points]** The joint pdf of random variables $\mathbb{X}$ and $\mathbb{Y}$ is

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} a, & 0 \le u \le 1, 0 \le v \le 1, u+v \le 1, \\ b, & 0 \le u \le 1, 0 \le v \le 1, 1 < u+v \le 2. \end{cases}$$

   What is the numerical value of $a + b$?

4. **[24 points]** $\mathbb{X}$ denotes a continuous random variable with pdf $f_{\mathbb{X}}(u)$ satisfying

$$f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u) \text{ for all } u, \ -\infty < u < \infty.$$

   Suppose that $\mathsf{var}(\mathbb{X}) = 9$. Let $\mathbb{Y} = |\mathbb{X}|$ and $\mathbb{Z} = -\mathbb{X}$, and consider the eight statements below for all random variables satisfying these conditions.

   Mark each statement as TRUE or FALSE.
   You do not need to provide any justification for your answers.

   Each correct answer will earn you 3 points. In order to discourage guessing, 3 points will be **deducted** from your score for each incorrect answer. Statements with both boxes unmarked will not affect your score. If you mark one box and then change your mind and mark the other, please be sure to indicate clearly what your final answer is.

   A net negative score on this problem will reduce your total exam score.

TRUE FALSE

□ □ $P\{\mathbb{X} > \alpha\} = F_{\mathbb{X}}(-\alpha)$ for all $\alpha, \; -\infty < \alpha < \infty$.

□ □ $F_{\mathbb{Y}}(v) = 2F_{\mathbb{X}}(v) - 1$ for $v \geq 0$, and 0 for $v < 0$.

□ □ $F_{\mathbb{Z}}(w) = F_{\mathbb{X}}(-w)$ for all $w, \; -\infty < w < \infty$.

□ □ $f_{\mathbb{Z}}(w) = f_{\mathbb{X}}(-w)$ for all $w, \; -\infty < w < \infty$.

□ □ $\mathsf{E}[\mathbb{Y}^2] = 9$.

□ □ $\mathsf{E}[\mathbb{Y}] = 3$.

□ □ $\mathsf{var}(\mathbb{Y}) < 9$.

□ □ $\mathbb{X}$ and $\mathbb{Y}$ are independent random variables.

5. **[18 points]** Let $\mathbb{X}$ denote the lifetime (measured in years) of a mobile phone drawn at random from a set of mobile phones that *appear* to be identical, but are not. For *half* the phones in the set, $\mathbb{X}$ is a *discrete* random variable taking on value 1 with probability $\frac{1}{3}$ and value 3 with probability $\frac{2}{3}$. The lifetimes of the other phones are *exponentially* distributed random variables with parameter λ.

   A phone is chosen at random from this set of mobile phones.

   (a) **[8 points]** Given that the chosen phone is working at the end of 4 years, what is the conditional probability that it will work for at least 3 more years?

   (b) **[10 points]** Given that the phone is working at the end of two years, what is the conditional probability that it will work for *at least three* more years?

6. **[22 points]** Let $\mathbb{X}$ denote a continuous random variable. When hypothesis H_0 is true, $\mathbb{X}$ is uniformly distributed on $[50, 100]$. When H_1 is true, $\mathbb{X}$ is uniformly distributed on $[0, 60]$. Recall that for a decision rule D, the *decision region* $\Gamma_{i,D}$ is the set of real numbers such that the decision is in favor of H_i whenever the observed value of $\mathbb{X}$ belongs to $\Gamma_{i,D}$.

   (a) **[5 points]** Find the decision region $\Gamma_{0,\text{ML}}$ for the *maximum-likelihood* (ML) decision rule.

   (b) **[5 points]** What is the value of $P_{\text{miss,ML}}$, the probability of *missed detection* (also known as theType II error probability), for the maximum-likelihood decision rule?

   (c) **[6 points]** If the *prior probabilities* π_0 and π_1 of the hypotheses are such that $\pi_1 = 2\pi_0$, what is $\Gamma_{1,\text{MAP}}$ for the *maximum a posteriori* (MAP) decision rule? Remember that the MAP decision rule is also the *minimum-error-probability* (MEP) decision rule.

   (d) **[6 points]** What is the (average) error probability $P_{e,\text{MAP}}$ of the MAP decision rule?

7. **[18 points]** Let $\mathbb{X}$ and $\mathbb{Y}$ be jointly continuous random variables with joint pdf

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \frac{1}{3} & 1 \le u \le 2,\ 2 \le v \le 3 \\ \frac{2}{3} & 3 \le u \le 5,\ 3 \le v \le \frac{7}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(a) **[6 points]** Are $\mathbb{X}$ and $\mathbb{Y}$ independent?

(b) **[6 points]** Find the marginal pdf $f_{\mathbb{X}}(u)$.
To obtain full credit, you must specify the value of $f_{\mathbb{X}}(u)$ for all $u, -\infty < u < \infty$.

(c) **[6 points]** Find $P\{\mathbb{Y} > \mathbb{X}\}$.