University of Illinois Fall 2013

ECE 313: Final Examination

Monday December 16, 2013   8:00 a.m. — 11:00 a.m.

1. (a)
   - TRUE: This is the law of total probability: $P(A \mid B)P(B) + P(A \mid B^c)P(B^c) = P(A)$.
   - TRUE: $P(A \mid B)P(B) + P(A^c \mid B)P(B) = P(AB) + P(A^cB) = P(B)$.
   - TRUE: $P(A \mid B^c)P(B^c) + P(A^c \mid B)P(B) = P(AB^c) + P(A^cB) = P(A \oplus B) = P(A \cup B) - P(A \cap B)$.

   (b)
   - FALSE: If $b > a$, then it is possible that $F_{\mathbb{X}}(b) = F_{\mathbb{X}}(a)$.
   - TRUE: The CDF is s nondecreasing function.
   - TRUE: The CDF is a continuous function increasing from 0 to 1.

   (c)
   - TRUE: The sum of independent $\mathsf{Binomial}(m,p)$ and $\mathsf{Binomial}(n,p)$ random variables is a $\mathsf{Binomial}(m+n,p)$ random variable.
   - FALSE: The sum of independent $\mathsf{Geometric}(p)$ random variables is a $\mathsf{NegativeBinomial}(2,p)$ random variable, not a $\mathsf{Geometric}(p)$ random variable
   - FALSE: The difference *is* a Gaussian random variable with mean $\mu_{\mathbb{X}} - \mu_{\mathbb{Y}}$ but its variance is $\sigma^2_{\mathbb{X}} + \sigma^2_{\mathbb{Y}}$, not $\sigma^2_{\mathbb{X}} - \sigma^2_{\mathbb{Y}}$.

   (d)
   - FALSE: Since $\mathsf{var}(\mathbb{X} \pm \mathbb{Y}) = \mathsf{var}(\mathbb{X}) + \mathsf{var}(\mathbb{Y}) \pm 2\mathsf{cov}(\mathbb{X},\mathbb{Y})$, all we can conclude is that $\mathsf{cov}(\mathbb{X},\mathbb{Y}) = 0$.
   - TRUE: If $\mathsf{var}(\mathbb{X}+\mathbb{Y}) = \mathsf{var}(\mathbb{X}-\mathbb{Y})$ then $\mathsf{cov}(\mathbb{X},\mathbb{Y}) = 0$ (cf. previous answer) and so $\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated.*
   - TRUE: $\mathsf{cov}(\mathbb{X}+\mathbb{Y},\mathbb{X}-\mathbb{Y}) = \mathsf{var}(\mathbb{X}) - \mathsf{var}(\mathbb{Y}) = 0$.
   - TRUE:

     $$\mathsf{var}(2\mathbb{X}+3\mathbb{Y}) = 4\cdot\mathsf{var}(\mathbb{X}) + 9\cdot\mathsf{var}(\mathbb{Y}) + 2\cdot 2\cdot 3\cdot\mathsf{cov}(\mathbb{X},\mathbb{Y})$$
     $$\mathsf{var}(3\mathbb{X}+2\mathbb{Y}) = 9\cdot\mathsf{var}(\mathbb{X}) + 4\cdot\mathsf{var}(\mathbb{Y}) + 2\cdot 3\cdot 2\cdot\mathsf{cov}(\mathbb{X},\mathbb{Y})$$

     Since these two variances are equal, we conclude that

     $$4\cdot\mathsf{var}(\mathbb{X}) + 9\cdot\mathsf{var}(\mathbb{Y}) = 9\cdot\mathsf{var}(\mathbb{X}) + 4\cdot\mathsf{var}(\mathbb{Y}) \Rightarrow \mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$$

   (e)
   - TRUE: Chebyshev's Inequality for continuous random variables tells us that $P\{|X| > 2\sigma\} = P\{|X| > 6\} \le \frac{1}{4}$. From the symmetry of the pdf, we know that $P\{X > 6\} = P\{X < -6\}$ and so $P\{\mathbb{X} > 6\} \le \frac{1}{8} \Rightarrow P\{X \le 6\} \ge \frac{7}{8} = 0.875$.
   - TRUE: The attached table shows that $P\{\mathbb{X} \le 6\} = \Phi(2) = 0.9772$.
   - FALSE: The support of a zero-mean uniformly distributed random variable with standard deviation 3 is $[-3\sqrt{3}, 3\sqrt{3}]$. Since $3\sqrt{3} < 3\sqrt{4} = 6$, it must be that $P\{\mathbb{X} \le 6\}$ *equals* 1.

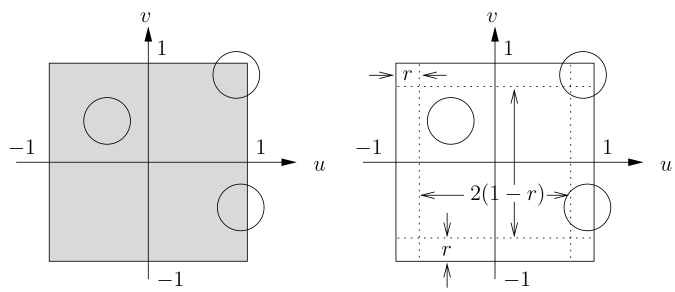
2. For 99% confidence we take $a = 10$, so the half-width of the confidence interval is $\frac{a}{2\sqrt{n}} = \frac{5}{\sqrt{n}}$, which should be ≤ 0.1. This requires $n \ge (\frac{5}{0.1})^2 = 2500$.

3. The pdf of $\mathbb{Y}$ is as shown below.

By inspection, $P\left\{|\mathbb{Y}| < \frac{1}{2}\right\} = \frac{1}{2}$ and $P\left\{\mathbb{Y} > 0 \middle| \mathbb{Y} < \frac{1}{2}\right\} = \frac{P\left\{0 < \mathbb{Y} < \frac{1}{2}\right\}}{P\left\{\mathbb{Y} < \frac{1}{2}\right\}} = \frac{1}{5}$.

Finally, $\mathsf{E}[\mathbb{Y}] = \int_{-1}^{0} v(1+v)\,dv + \int_{0}^{1} v^2\,dv = \frac{v^2}{2} + \frac{v^3}{3}\bigg|_{-1}^{0} + \frac{v^3}{3}\bigg|_{0}^{1} = \frac{-1}{6} + \frac{1}{3} = \frac{1}{6}$.

4. The joint pdf has value 4 on the shaded region.



Now, $\mathbb{Z} = 2$ or 1 or 0 according as $(\mathbb{X}, \mathbb{Y})$ is respectively in one of the four $r \times r$ corner squares, or one of the four $r \times 2(1-r)$ edge rectangles, or the $2(1-r) \times 2(1-r)$ central square shown above in the sketch on the right. Hence, $P(\mathbb{Z} = 2) = r^2, P(\mathbb{Z} = 1) = 2r(1-r), P(\mathbb{Z} = 0) = (1-r)^2$, that is, $\mathbb{Z} \sim \mathsf{Binomial}(2, r)$.

The next step is easy: Since $\mathsf{E}[\mathbb{Z}] = 2r = \frac{3}{2}$, we get $r = \frac{3}{4}$.

5.
$$F_W(c) = P(W \le c) = P(X \le cX + cY) = P\left(Y \ge \frac{1-c}{c}X\right).$$

For $c < 0.5$, the line $Y = \frac{1-c}{c}X$ intersects the square $[0,1]^2$ at $(\frac{c}{1-c}, 1)$, hence

$$P(W \le c) = \frac{c}{2(1-c)}.$$

For $c \ge 0.5$, the line $Y = \frac{1-c}{c}X$ intersects the square $[0,1]^2$ at $(1, \frac{1-c}{c})$, hence

$$P(W \le c) = 1 - \frac{1-c}{2c}.$$

Together,

$$F_W(c) = \begin{cases} \frac{c}{2(1-c)} & \text{if } c < 0.5 \\ 1 - \frac{1-c}{2c} & \text{if } c \ge 0.5. \end{cases}$$

6. (a) Let $H3$ denote the probability of heads showing up on all three flips. Let F denote the event that the fair coin is chosen, and B denote the event that the biased coin is chosen.

$$P(H3) = P(H3|F)P(F) + P(H3|B)P(B) = \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3 \times \left(\frac{1}{2}\right) = \frac{35}{128}.$$

(b) Let $H2$ denote the probability of heads showing up on all three flips. Let F denote the event that the fair coin is chosen, and B denote the event that the biased coin is chosen.

$$P(H2) = P(H2|F)P(F) + P(H2|B)P(B) = 3\times\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right) + 3\times\left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right) \times \left(\frac{1}{2}\right) = \frac{51}{128}.$$

$$P(F|H2) = \frac{P(H2|F)P(F)}{P(H2)} = \frac{3\times\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)}{\frac{51}{128}} = \frac{\frac{3}{16}}{\frac{51}{128}} = \frac{8}{17}.$$

7. (a) X and Y are *not* independent because the support is not a product set.

   (b) Recall that $p_X(i) = \sum_{j=-\infty}^{\infty} p_{X,Y}(i,j)$.
   For $1 \leq i$,

$$p_X(i) = \sum_{j=i+1}^{\infty} cp^2(1-p)^{j-2} \; = \; cp(1-p)^{i-1} \sum_{j=i+1}^{\infty} p(1-p)^{j-1-i} \; = \; cp(1-p)^{i-1}.$$

   For all other i, $p_X(i) = 0$. Hence, $X \sim Geometric(p)$

   (c) The event $\{2Y^2 - X \geq 0\} = \{2Y^2 \geq X\} = \Omega$ because the support of the joint pmf is $j \geq i+1$. Hence, $P\{2Y^2 - X \geq 0\} = 1$.

8. Think of 9 stones (to account for the total of 9) laid down on a line. We place two sticks in between the stones to split the allocations of 9 stones into three draws. There are 8 possible places for the two sticks – so the total number of choices is $\binom{8}{2} = 28$.

9. It is easier to compute the probability that you will not get a U. Beginning with two U's and seven letters which are not U's, we get the probability of not drawing a U as: $\frac{7}{9} \cdot \frac{6}{8} \cdot \frac{5}{7} \cdot \frac{4}{6} = \frac{5}{18}$. This means that with probability $\frac{13}{18}$ you would draw a U.

10. (a) $\mathrm{Var}(X + 2Y + 3Z + 5) = \mathrm{Var}(X) + \mathrm{Var}(2Y) + \mathrm{Var}(3Z) = 9 + 36 + 81 = 126$.

    (b) $\mathrm{Cov}(X+Y, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Y) + \mathrm{Cov}(Y,Z) = \mathrm{Cov}(Y,Y)$ $= \mathrm{Var}(Y) = 9$.

    (c) $E[X^2Y^2] = E[X^2]E[Y^2] = (9+16)^2 = 625$.

    (d) $\mu_U = \mu_V = 8$. Further, $\sigma_U^2 = \sigma_V^2 = 18$. Now $\mathrm{Cov}(U,V) = \mathrm{Var}(Y) = 9$. So $\rho_{U,V} = \frac{1}{2}$. Thus the linear minimum mean square estimator of U given V = 10 is equal to $8 + \frac{1}{2}(10-8)$ $= 9$.