

**ECE 313 (Section B)**  
**Quiz 1 - Tuesday, November 12**

Name and NetID: \_\_\_\_\_

**Problem 1** - From past experience a professor knows that the test score of a student taking her final examination is a random variable with a mean of 75 and standard deviation of 8. How many students would have to take the examination to ensure, with probability at least .95, that the class average would be at least 73?  $\rightarrow \text{Var}(X) = 64$

**Hint:** Remember the central limit theorem example solved in the class for the average of a sequence of independently, identically distributed, random variables. Assume that all the test score of all the students taking the final examination has the same mean and standard deviation.

$$\begin{aligned} * \quad & \left\{ \begin{aligned} \bar{X} &= \frac{X_1 + X_2 + \dots + X_n}{n} \\ E[\bar{X}] &= \frac{1}{n} E[X_1] + E[X_2] + \dots + E[X_n] = E[X_1] = 75 \\ \text{Var}(\bar{X}) &= \text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{1}{n^2} [\text{Var}(X_1) + \dots + \text{Var}(X_n)] = \frac{\text{Var}(X_1)}{n} = \frac{64}{n} \end{aligned} \right. \end{aligned}$$

$$\text{CLT} \Rightarrow \bar{X} \sim N\left(75, \frac{8}{\sqrt{n}}\right)$$

$$P(\bar{X} \geq 73) = P\left(\frac{\bar{X} - 75}{\frac{8}{\sqrt{n}}} \geq \frac{73 - 75}{\frac{8}{\sqrt{n}}}\right) = P\left(Z \geq \frac{-\sqrt{n}}{4}\right) \geq 0.95$$

$$1 - \Phi\left(-\frac{\sqrt{n}}{4}\right) \geq 0.95 \Rightarrow \Phi\left(-\frac{\sqrt{n}}{4}\right) \geq 0.95$$

$$* \quad 1 - \Phi\left(\frac{\sqrt{n}}{4}\right)$$

$$\frac{\sqrt{n}}{4} \geq 1.65 \Rightarrow \sqrt{n} \geq 6.6$$

Needs table  $n \geq (6.6)^2 = 43.56$

$$n \text{ is an integer} \Rightarrow \boxed{n = 44}$$

We need at least 44 students taking the exam.

**Problem 2** - Suppose  $n$  fair dice are independently rolled. Let:

$$X_k = \begin{cases} 1 & \text{if one shows on the } k^{\text{th}} \text{ die} \\ 0 & \text{else} \end{cases}$$

$$Y_k = \begin{cases} 1 & \text{if two shows on the } k^{\text{th}} \text{ die} \\ 0 & \text{else} \end{cases}$$

Let  $X = \sum_{k=1}^n X_k$ , which is the number of one's showing, and  $Y = \sum_{k=1}^n Y_k$ , which is the number of two's showing. Note that if a histogram is made recording the number of occurrences of each of the six numbers, then  $X$  and  $Y$  are the heights of the first two entries in the histogram.

- Find  $E[X_1]$  and  $\text{Var}(X_1)$ .
- Find  $E[X]$  and  $\text{Var}(X)$ .
- Find  $\text{Cov}(X_i, Y_j)$  if  $1 \leq i \leq n$  and  $1 \leq j \leq n$  (Hint: Does it make a difference if  $i = j$ ?)
- Find  $\text{Cov}(X, Y)$ .
- Find the correlation coefficient  $\rho_{X,Y}$ . Are  $X$  and  $Y$  positively correlated, uncorrelated, or negatively correlated?  $\rho_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-\frac{n}{36}}{\frac{5n}{36}} = -\frac{1}{5} \Rightarrow \text{Negatively correlated}$

Hint: Remember that:

$$\text{Cov}(X, Y) = \text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n Y_j\right) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j)$$

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- a)  $X_k$  is bernouli random variable with  $p = \frac{1}{6}$  (Probability of one showing)

$$E[X_k] = 1(p) + 0(1-p) = \frac{1}{6}$$

$$\text{Var}(X_k) = E[X_k^2] - E[X_k]^2 = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$$

$$E[X_k^2] = (1)^2 p + (0)^2 (1-p) = \frac{1}{6}$$

b)  $E[X] = E\left[\sum_{k=1}^n X_k\right] = E[X_1] + E[X_2] + \dots + E[X_n] = \frac{n}{6}$

$$\text{Var}(X) = \text{Var}\left(\sum_{k=1}^n X_k\right) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n) = \frac{5n}{36}$$

- c) if  $i \neq j$  then  $X_i$  &  $Y_j$  are independent (different dices)

$$\Rightarrow \text{Cov}(X_i, Y_j) = 0$$

if  $i = j$  then:  $\text{Cov}(X_i, Y_i) = E[X_i Y_i] - E[X_i] E[Y_i]$

$X_i Y_i \rightarrow$  joint PMF

$(0,0)$	$\frac{4}{6}$
$(0,1)$	$\frac{1}{6}$
$(1,0)$	$\frac{1}{6}$
not possible (1,1)	0

$$\Rightarrow E[X_i Y_i] = \sum_{i=0,1} \sum_{j=0,1} ij P_{ij} = 0 \times 0 \left(\frac{4}{6}\right) + 0 \times 1 \left(\frac{1}{6}\right) + 1 \times 0 \left(\frac{1}{6}\right) = 0$$

$$\text{Cov}(X_i, Y_i) = -\frac{1}{36}$$

d)  $\text{Cov}(X, Y) = \text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n Y_j\right) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j) = \sum_{i=1}^n \text{Cov}(X_i, Y_i) + \sum_{i \neq j} \text{Cov}(X_i, Y_j)$

$$\text{Cov}(X, Y) = -\frac{n}{36}$$