ECE 313 - Section B
Midterm Exam
Fall 2013

Name: Solution

NetID: _____

- **Be sure that your exam booklet has 10 pages.**
- **Write your name at the top of each page.**
- This is a **closed book** exam.
- You may consult both sides of your 8.5" x 11" sheet of notes.
- No calculators, cell phones, PDAs, tablets, or laptop computers are allowed.
- **Please show all your work.** Answers without appropriate justification will receive **very little** or **no credit**.
- If you need extra space, use the back of the previous page.

Problem 1	_____	(25 pts)
Problem 2	_____	(15 pts)
Problem 3	_____	(15 pts)
Problem 4	_____	(25 pts)
Problem 5	_____	(20 pts)
TOTAL	_____	(100 pts)

Problem 1 (25 pts) – For each of the following parts, provide a short answer in the space provided, or choose the statement that is TRUE. Show your work and a justification for your answer to get partial credit.

Homework 1 Problem 1 Part C

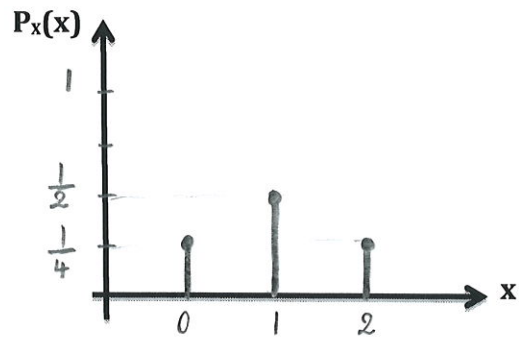
Part A (2 pts): We pick a random number from the set of odd numbers with distinct digits between 1000 and 10000. What is the size of sample space in this experiment?

- There are four positions to fill : units, tens, hundreds, and thousands.
- We have 5 options for the units : $\{1, 3, 5, 7, 9\}$
- Thousands position cannot be a zero
- We cannot use the numbers used in other positions

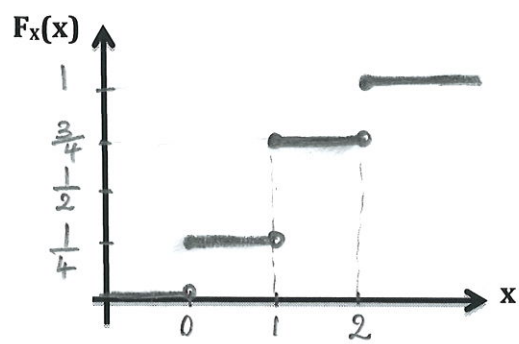
$\underline{8} \times \underline{8} \times \underline{7} \times \underline{5} = \boxed{2240}$

Homework 5 Problem 2 Parts a, b

Part B (5 pts): A fair coin is tossed two times. X is defined as the total number of heads showing up in this experiment. Draw the PMF and CDF of X. Show the limits on x and y axes appropriately.



PMF



CDF

⇒ Sample Space:

$S = \{HH, HT, TH, TT\}$

⇒ X can take any of the values $\{0, 1, 2\}$:

$X = 0 \rightarrow \{TT\}$
$X = 1 \rightarrow \{HT, TH\}$
$X = 2 \rightarrow \{HH\}$

⇒ PMF can be described as follows :

$P(X=0) = P\{TT\} = \frac{1}{4}$

$P(X=1) = P\{HT, TH\} = \frac{2}{4} = \frac{1}{2}$

$P(X=2) = P\{HH\} = \frac{1}{4}$

⇒ CDF can be computed by : $F(a) = \sum_{\text{all } x_i \leq a} p(X = x_i)$

$$F(a) = \begin{cases} 0 & ; a < 0 \\ \frac{1}{4} & ; 0 \leq a < 1 \\ \frac{3}{4} & ; 1 \leq a < 2 \\ 1 & ; 2 \leq a \end{cases}$$

Problem 1, continued:

Lecture 4 Slide 13 + In-class Project 2 Problem 1

Part C (5 pts): Consider the experiment of tossing two dice. Events A, B, C, and D are defined as below:

A = "first die results in a 1, 2, or 3."
B = "second die results in a 4, 5, or 6."
C = "first die results in a 3, 4, or 5."
D = "the sum of the two faces is 7."

Which of the following statements are TRUE?

1 – A and C are pairwise independent.
2 – A and B are pairwise independent. → TRUE
3 – A and B are mutually exclusive.
4 – A, B, and D are pairwise independent. → TRUE
5 – A, B, and D are mutually independent.

$A = \{(1,1), (1,2), \dots, (1,6), (2,1), (2,2), \dots, (2,6), (3,1), (3,2), \dots, (3,6)\}$

$B = \{(1,4), (2,4), \dots, (6,4), (1,5), (2,5), \dots, (6,5), (1,6), (2,6), \dots, (6,6)\}$

$C = \{(3,1), (3,2), \dots, (3,6), (4,1), (4,2), \dots, (4,6), (5,1), (5,2), \dots, (5,6)\}$

$D = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

× 1 $A \cap C = \{(3,1), (3,2), \dots, (3,6)\} \Rightarrow P(A \cap C) = \frac{1}{6} \neq P(A) \cdot P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$A \cap B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

× 3 $A \cap B \neq \phi$ ✓ 2 $\Rightarrow P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

4 $A \cap D = \{(1,6), (2,5), (3,4)\} \rightarrow P(A \cap D) = \frac{1}{12} = P(A) \cdot P(D) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ ✓

$B \cap D = \{(3,4), (2,5), (1,6)\} \rightarrow P(B \cap D) = \frac{1}{12} = P(B) \cdot P(D) = \frac{1}{12}$ ✓

5 $A \cap B \cap D = A \cap D = B \cap D \rightarrow P(A \cap B \cap D) = \frac{1}{12} \neq P(A)\,P(B)\,P(D) = \frac{1}{24}$ ×

Lecture 16 Slides 16-17 + Lecture 10 Slide 14

Part D (3 pts): X is a normally distributed random variable with parameters (μ, σ). Random variable Y is defined as $Y = aX + b$. Determine the mean, the variance, and the PDF of Y:

E[Y] = $a\mu + b$ Var(Y) = $a^2\sigma^2$

$f_Y(y) = \frac{1}{a\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y - a\mu - b}{a\sigma}\right)^2\right]$

Y is also normally distributed.

PDF of Normal generally: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

Problem 1, continued:

Lecture 10 slide 13

Part E (2 pts): With Φ(x) being the probability that a normal random variable with mean 0 and variance 1 is less than *x*, which of the following expressions are TRUE?

- Φ(-x) = Φ(x)
- $\boxed{\Phi(x) + \Phi(-x) = 1}$
- Φ(-x) = 1/ Φ(x)

$Q(u) = 1 - \Phi(u) = \Phi(-u) \leftarrow \Phi(x) = F_Z(x) = P(Z < x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Part E (8 pts): For each of the following random variables, determine if it is discrete or continuous, identify the distribution and its parameters, and then write the formula for probability mass function (pmf) or probability distribution function (pdf) based on the information provided:

Homework 4 Problem 6

I. The number of jobs arriving at a file server in an interval of 1 second is Poisson distributed with an average arrival rate of 0.1 per second. Let ***Y*** be the inter-arrival time of the jobs.

$Y \sim$ Exponential $(\lambda = 0.1)$

$$f(y) = \begin{cases} 0.1\, e^{-0.1y} & , y > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Homework 3 Problem 3 Part b

II. A certain manufacturing company produces chipsets that are not defective with probability of $p = 0.85$. A quality control crew randomly picks chipsets and tests them for defects. Let ***N*** be the number of trials until the first defective chipset is found.

$N \sim$ Geometric $(p = 0.15)$

$P(\text{Defective}) = 1 - p$
$= 0.15$

$P\{N = n\} = (1 - 0.15)^{n-1}(0.15) = (0.85)^{n-1}\, 0.15$
$n = 1, 2, \ldots$

Lecture 6 slide 3 + Homework 1 Problem 1

Problem 1 (Part E), continued:

III. Consider the following program segment consisting of a 'for' loop. Assume that the Boolean expression *S* is true with probability *p* and false with probability *1-p*. If the successive tests on *S* are independent from each other, let the random variable ***X*** be the number of times that statement A is executed.

```
for (int i = 0; i < 25; i++)
{
      if (S == TRUE)
            A;
      else
            B;
}
```

$X \sim \text{Binomial}(n = 25, p)$

$P(X = k) = \binom{25}{k} p^k (1-p)^{25-k}$

Lecture 11 slide 6 + In-class Project 3 + Homework 5 Problem 5

IV. ***T*** is the lifetime of a switch in a transaction clearing system and has a memory-less property. The mean time to failure of the system is 4 years.

T is contineous random variable with memoryless property $\Rightarrow$ Exponential

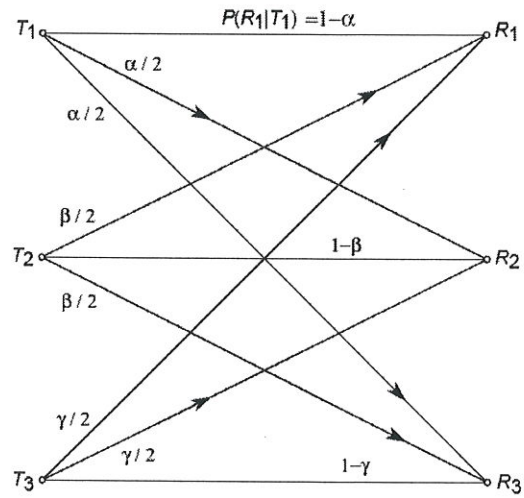
Mean Time to Failure $= E[T] = \frac{1}{\lambda} = 4 \Rightarrow \lambda = \frac{1}{4} = 0.25$

$\Rightarrow T \sim \text{Exponential}\left(\lambda = \frac{1}{4}\right)$

$$f(t) = \begin{cases} \frac{1}{4} e^{-\frac{t}{4}}, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Homework 1
Problem 5

Problem 2 (15 pts) – Consider a trinary communication channel whose channel diagram is shown in the following figure. For i = 1, 2, 3 let T_i denote the event "digit i is transmitted" and let R_i denote the event "digit i is received".



Assume that digit 3 is transmitted 3 times more frequently than digit 1, and digit 2 is sent twice as often as digit 1. Events T_1, T_2, and T_3 are mutually exclusive and collectively exhaustive.

Part A (5 pts): Calculate the probabilities of T_1, T_2, and T_3.

$$\begin{cases} 3P(T_1) = P(T_3) \\ 2P(T_1) = P(T_2) \\ P(T_1) + P(T_2) + P(T_3) = 1 \end{cases}$$

← Mutually exclusive & collectively exhaustive

$$\Rightarrow \quad P(T_1) = \frac{1}{6}, \quad P(T_2) = \frac{1}{3}, \quad P(T_3) = \frac{1}{2}$$

Part B (5 pts): What is the expression for the conditional probability that given a 1 is received, a 1 was transmitted?

$$P(T_1 | R_1) = \frac{P(R_1 | T_1) P(T_1)}{P(R_1)}$$

$$P(R_1 | T_1) = 1-\alpha$$

$$P(R_1) = P(R_1 | T_1) P(T_1) + P(R_1 | T_2) P(T_2) + P(R_1 | T_3) P(T_3)$$

$$= (1-\alpha)\left(\frac{1}{6}\right) + \left(\frac{\beta}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{\gamma}{2}\right)\left(\frac{1}{2}\right) = \frac{2(1-\alpha) + 2\beta + 3\gamma}{12}$$

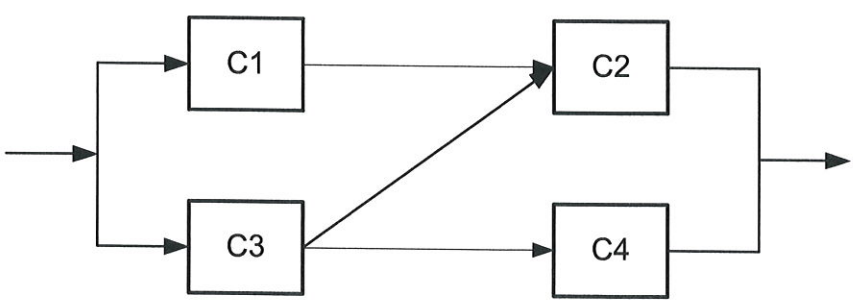
$$\Rightarrow \quad P(T_1 | R_1) = \frac{2(1-\alpha)}{2(1-\alpha) + 2\beta + 3\gamma}$$

Problem 2, continued:

Part C (5 pts): Derive an expression for the probability of transmission error. (Hint: Use the probability of success. Success is defined as a digit *i* is transmitted and it is received.)

$$P(\text{error}) = 1 - P(\text{Success})$$
$$= 1 - P(R_1|T_1)P(T_1) - P(R_2|T_2)P(T_2) - P(R_3|T_3)P(T_3)$$
$$= 1 - (1-\alpha)\frac{1}{6} - \frac{1}{3}(1-\beta) - \frac{1}{2}(1-\gamma)$$
$$P(\text{error}) = \frac{\alpha + 2\beta - 3\gamma}{6}$$

Problem 3 (15 pts) – Consider the non-series-parallel system of four independent components shown in the following figure. The system is considered to be functioning properly if all components along at least one path from input to output are functioning properly.

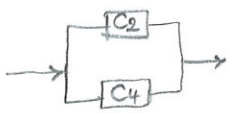


Lecture 5 slide 18

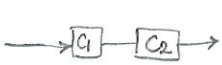
Part A (5 pts): Determine an expression for the system reliability as a function of component reliabilities (Assume that the reliability of each component is equal to *R*).

We condition on component C_3:

$$P(X) = P(X|X_3)\underbrace{P(X_3)}_{C_3 \text{ Working}} + P(X|\overline{X}_3)\underbrace{P(\overline{X}_3)}_{C_3 \text{ Not working}}$$



$$P(X|X_3) = 1 - (1-R_2)(1-R_4)$$



$$P(X|\overline{X}_3) = R_1 R_2$$
$$P(X_3) = R_3$$
$$P(\overline{X}_3) = (1-R_3)$$

$$\Rightarrow R_{sys} = R_3 \cdot [1-(1-R_2)(1-R_4)] + (1-R_3)\cdot R_1 R_2$$
$$= R[1-(1-R)^2] + (1-R)R^2$$
$$\boxed{R_{sys} = 3R^2 - 2R^3}$$

Problem 3, continued:

In-class Prj 3

Part B (5 pts): Calculate the mean time to the failure (MTTF) of the system. Assume that the time to failure (lifetime) of each component is exponentially distributed with parameter λ.

$$R_{sys}(t) = 3R^2(t) - 2R^3(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$MTTF = E[T] = \int_0^\infty R_{sys}(t)\,dt = \int_0^\infty 3e^{-2\lambda t} - 2e^{-3\lambda t}\,dt$$

$$\Rightarrow MTTF = \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6\lambda}$$

Part C (5 pts): Recall the reliability expression for a TMR system (with a perfect voter) from your in-class project. Compare MTTF of the system shown here with MTTF of a TMR system composed of the same components (with reliability of *R* and an exponential time to failure with parameter λ). Which one will fail on average earlier?

$$R_{TMR} = 3R^2 - 2R^3 = R_{sys} \Rightarrow MTTF_{TMR} = MTTF_{sys} = \frac{5}{6\lambda}$$

Homework 4 Problem 2 + Homework 5 Problem 4 + Homework 6 Problem 2

Name: _____

Problem 4 (25 pts) – Let the probability density of X be given by:

$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2), & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

Part A (5 pts): Find the cumulative distribution function (CDF) of X.

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_0^t \frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left(2x^2 - \frac{2x^3}{3}\right)\Big|_0^t$$

$$\boxed{F_X(x) = \frac{3}{4}x^2 - \frac{1}{4}x^3;\ 0 < x < 2}$$

Part B (3 pts): Find $P(1 < X < 2)$.

$$P(1 < X < 2) = F_X(2) - F_X(1) = \left[\frac{3}{4}\overset{3}{(2)^2} - \frac{1}{4}\overset{2}{(2)^3}\right] - \left[\cancel{\frac{3}{4} - \frac{1}{4}}\ \frac{1}{2}\right] = \boxed{\frac{1}{2}}$$

Part C (5 pts): Calculate the expected value of X.

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x\,\frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[4\frac{x^3}{3} - \frac{2x^4}{4}\right]\Big|_0^2$$

$$= \frac{x^3}{2} - \frac{3}{16}x^4\Big|_0^2 = \frac{8}{2} - \frac{3}{16}16 = \boxed{1}$$

Part D (6 pts): Calculate the variance of X.

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^2 x^2\,\frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[\frac{4x^4}{4} - 2\frac{x^5}{5}\right]\Big|_0^2$$

$$= \frac{3}{8}\left[\overset{2}{\cancel{16}} - \frac{2}{5}\overset{4}{\cancel{32}}\right] = 6 - \frac{24}{5} = \frac{6}{5}$$

$$\boxed{Var(X) = \frac{6}{5} - 1 = \frac{1}{5}}$$

Part E (6 pts): Let $Y = \sqrt{X + 1}$. Find the probability distribution function (PDF) of Y.

$$Y = \sqrt{X+1} \qquad 0 < x < 2 \Rightarrow 1 < y < \sqrt{3}$$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X+1} \leq y) = P(X+1 \leq y^2)$$

$$= P(X \leq y^2 - 1) = F_X(y^2 - 1)$$

From A $\longrightarrow$ $F_X(x) = \frac{3}{4}x^2 - \frac{1}{4}x^3$

$$\Rightarrow F_Y(y) = F_X(y^2 - 1) = \frac{3}{4}(y^2-1)^2 - \frac{1}{4}(y^2-1)^3$$

PDF of Y $\Rightarrow$ $f_Y(y) = \frac{dF_Y}{dy} = \frac{3}{4}4y(y^2-1) - \frac{1}{4}6y(y^2-1)^2, \quad 1 < y < \sqrt{3}$

Homework 6
Problem 6

Problem 5 (20 pts) – Let X and Y be two continuous random variables with joint density function:

$$f(x,y) = \begin{cases} cy + \frac{1}{9}, & 0 < x < 3\ , x < y < 3 \\ 0, & otherwise \end{cases}$$

where c is a constant.

Part A (5 pts): What is the value of c?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^3\int_x^3 cy + \frac{1}{9}\,dy\,dx = \int_0^3 \underbrace{\frac{cy^2}{2} + \frac{1}{9}y}\Big|_x^3$$

$$\frac{9}{2}c + \frac{1}{3} - \frac{c}{2}x^2 - \frac{1}{9}x$$

$$= \frac{9}{2}cx + \frac{x}{3} - \frac{c}{6}x^3 - \frac{1}{18}x^2\Big|_0^3 = 1$$

$$\frac{27}{2}c + \cancel{1} - \frac{27}{6}c - \frac{1}{2} = \cancel{1} \Rightarrow \frac{2\times 27}{6}c = \frac{1}{2}$$

$$\boxed{c = \frac{1}{18}}$$

Part B (6 pts): Find the marginal pdfs of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^3 \frac{1}{18}y + \frac{1}{9}\,dy = \frac{y^2}{36} + \frac{1}{9}y\Big|_x^3$$

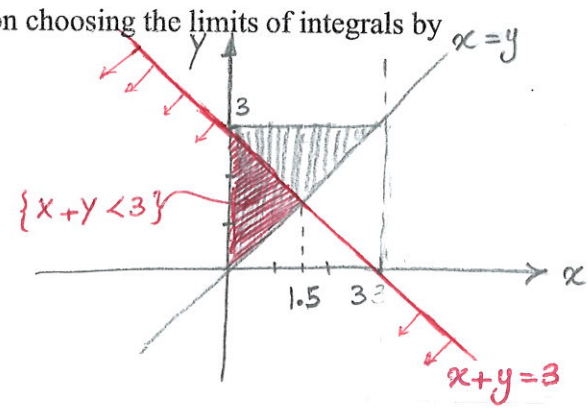
$$\boxed{f_X(x) = \frac{1}{4} + \frac{1}{3} - \frac{x^2}{36} - \frac{x}{9} = \frac{7}{12} - \frac{x^2}{36} - \frac{x}{9}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{18}y + \frac{1}{9}\,dx = \frac{1}{18}yx + \frac{1}{9}x\Big|_0^y = \boxed{\frac{1}{18}y^2 + \frac{1}{9}y}$$

Part C (2 pts): Are X and Y independent? Why?

No, because $f_{XY}(x,y) \neq f_X(x)\cdot f_Y(y)$

Part D (7 pts): Find $P\{X+Y<3\}$. Show your work on choosing the limits of integrals by marking the region of integral.



$$P\{X+Y<3\} = \int_0^{1.5}\int_x^{3-x} \frac{1}{18}y + \frac{1}{9}\,dy\,dx$$

$$= \int_0^{1.5} \frac{y^2}{36} + \frac{y}{9}\Big|_x^{3-x} = \int_0^{1.5}\left[\frac{(3-x)^2}{36} + \frac{3-x}{9} - \frac{x^2}{36} - \frac{x}{9}\right]dx$$

$$\int_0^{1.5} \frac{9 - 6x + \cancel{x^2} + 12 - 4x - \cancel{x^2} - 4x}{36}\,dx = \int_0^{1.5} \frac{21}{36} - \frac{14}{36}x\,dx$$

$$= \frac{21}{36}(1.5) - \frac{14}{36}\frac{(1.5)^2}{2} = 0.4375$$