University of Illinois **Fall 2012**

ECE 313: Problem Set 8: Problems and Solutions
Moments of jointly distributed random variables, minimum mean square error estimation

Due: Wednesday December 5 at 4 p.m.
Reading: *313 Course Notes* Sections 4.8-4.9

1. **[Covariance I]**
   Consider random variables X and Y on the same probability space.

   (a) If $\mathrm{Var}(X+2Y) = 40$ and $\mathrm{Var}(X-2Y) = 20$, what is $\mathrm{Cov}(X,Y)$?
   Solution:

$$\begin{aligned}\mathrm{Var}(X+2Y) &= \mathrm{Cov}(X+2Y, X+2Y) \\ &= \mathrm{Var}(X) + 4\mathrm{Var}(Y) + 4\mathrm{Cov}(X,Y) = 40\end{aligned}$$

   Similarly, $\mathrm{Var}(X-2Y) = \mathrm{Cov}(X-2Y, X-2Y) = \mathrm{Var}(X)+4\mathrm{Var}(Y)-4\mathrm{Cov}(X,Y) = 20$. Taking the difference of the two equations describing $\mathrm{Var}(X+2Y)$ and $\mathrm{Var}(X-2Y)$ yields $\mathrm{Cov}(X,Y) = 2.5$.

   (b) In part (a), determine $\rho_{X,Y}$ if $\mathrm{Var}(X) = 2 \cdot \mathrm{Var}(Y)$.
   Solution: Adding the two equations describing $\mathrm{Var}(X+2Y)$ and $\mathrm{Var}(X-2Y)$, we get

$$\begin{aligned}2\mathrm{Var}(X) + 8\mathrm{Var}(Y) &= 60 \\ 12\mathrm{Var}(Y) = 60\end{aligned}$$

   Hence, $\mathrm{Var}(Y) = 5$, $\mathrm{Var}(X) = 10$, and

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = 0.3536$$

2. **[Covariance II]**
   Suppose X and Y are random variables on some probability space.

   (a) If $\mathrm{Var}(X+2Y) = \mathrm{Var}(X-2Y)$, are X and Y uncorrelated ?
   Solution: Expanding each side of $\mathrm{Var}(X+2Y) = \mathrm{Var}(X-2Y)$ yields:
   $\mathrm{Var}(X) + 4\mathrm{Cov}(X,Y) + 4\mathrm{Var}(Y) = \mathrm{Var}(X) - 4\mathrm{Cov}(X,Y) + 4\mathrm{Var}(Y)$ implying that $\mathrm{Cov}(X,Y) = 0$. Hence, X and Y are uncorrelated.

   (b) If $\mathrm{Var}(X) = \mathrm{Var}(Y)$, are X and Y uncorrelated?
   Solution: No. The condition $\mathrm{Var}(X) = \mathrm{Var}(Y)$ does not imply that $\mathrm{Cov}(X,Y) = 0$.

3. **[Covariance III]**
   Rewrite the expressions below in terms of $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, $\mathrm{Var}(Z)$, and $\mathrm{Cov}(X,Y)$.

   (a) $\mathrm{Cov}(3X+2, 5Y-1)$
   Solution: $\mathrm{Cov}(3X+2, 5Y-1) = \mathrm{Cov}(3X, 5Y) = 15\mathrm{Cov}(X,Y)$.

(b) $\text{Cov}(2X+1, X+5Y-1)$.
Solution:

$$\begin{aligned}\text{Cov}(2X+1, X+5Y-1) &= \text{Cov}(2X, X+5Y) = \text{Cov}(2X, X) + \text{Cov}(2X, 5Y) \\ &= 2\text{Cov}(X, X) + 10\text{Cov}(X, Y) = 2\text{Var}(X) + 10\text{Cov}(X, Y)\end{aligned}$$

(c) $\text{Cov}(2X+3Z, Y+2Z)$ where Z is uncorrelated to both X and Y.
Solution:

$$\begin{aligned}\text{Cov}(2X+3Z, Y+2Z) &= \text{Cov}(2X, Y) + \text{Cov}(2X+2Z) + \text{Cov}(3Z, Y) + \text{Cov}(3Z, 2Z) \\ &= 2\text{Cov}(X, Y) + 4\text{Cov}(X, Z) + 3\text{Cov}(Z, Y) + 6\text{Cov}(Z, Z) \\ &= 2\text{Cov}(X, Y) + 6\text{Var}(Z)\end{aligned}$$

4. **[Covariance IV]**
Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The *signal-to-noise-ratio* (SNR) of the observation X_1 or X_2 is defined as the ratio $SNR_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.

(a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.
Solution: In general, for $S = \frac{X_1+X_2}{2}$.

$$\begin{aligned}E[S] &= \mu_S = E\left[\frac{X_1+X_2}{2}\right] = \mu \\ \sigma_S^2 &= \frac{\text{Var}(X_1+X_2)}{4} = \frac{2\sigma^2 + 2\text{Cov}(X_1, X_2)}{4} = \frac{\sigma^2 + \text{Cov}(X_1, X_2)}{2} \\ SNR_S &= \frac{2\mu^2}{\sigma^2 + \text{Cov}(X_1, X_2)}\end{aligned}$$

Thus, if X_1 and X_2 are uncorrelated, $SNR_S = \frac{2\mu^2}{\sigma^2} = 2SNR_X$. Thus, averaging improves the SNR by a factor equal to the number of observations being averaged, if the observations are uncorrelated.

(b) The system designer notices that the averaging strategy is giving $SNR_S = (1.5)SNR_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient $\rho_{X_1X_2}$.
Solution: Since $\text{Cov}(X_1, X_2) = \sigma^2\rho_{X_1,X_2}$, the formula above for SNR_S is equivalent to

$$SNR_S = \frac{2\mu^2}{\sigma^2(1+\rho_{X_1X_2})}.$$

Setting SNR_S equal to $1.5\frac{\mu^2}{\sigma^2}$ yields $\rho_{XY} = \frac{1}{3}$.

(c) Under what condition on $\rho_{X,Y}$ can the averaging strategy result in an SNR_S that is arbitrarily high?
Solution: $SNR_S \to \infty$ as $\rho_{X_1X_2} \to -1$.

5. **[Linear minimum MSE estimation from uncorrelated observations]**
Suppose Y is estimated by a linear estimator, $L(X_1, X_2) = a + bX_1 + cX_2$, such that X_1 and X_2 have mean zero and are uncorrelated with each other.

   (a) Determine a, b and c to minimize the MSE, $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of $E[Y]$, the variances of X_1 and X_2, and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.
   Solution: The MSE can be written as $E[((Y - bX_1 - cX_2) - a)^2]$, which is the same as the MSE for estimation of $Y - bX_1 - cX_2$ by the constant a. The optimal choice of a is $E[Y - bX_1 - cX_2] = E[Y]$. Substituting $a = E[Y]$, the MSE satisfies

$$\begin{aligned}
\text{MSE} &= \text{Var}(Y - bX_1 - cX_2) \\
&= \text{Cov}(Y - bX_1 - cX_2, Y - bX_1 - cX_2) \\
&= \text{Cov}(Y, Y) + b^2\text{Cov}(X_1, X_1) - 2b\text{Cov}(Y, X_1) + c^2\text{Cov}(X_2, X_2) - 2c\text{Cov}(Y, X_2) \\
&= \text{Var}(Y) + \left(b^2\text{Var}(X_1) - 2b\text{Cov}(Y, X_1)\right) + \left(c^2\text{Var}(X_2) - 2c\text{Cov}(Y, X_2)\right). \qquad (1)
\end{aligned}$$

   The MSE is quadratic in b and c and the minimizers are easily found to be $b = \frac{\text{Cov}(Y,X_1)}{\text{Var}(X_1)}$ and $c = \frac{\text{Cov}(Y,X_2)}{\text{Var}(X_2)}$. Thus, $L(X_1, X_2) = E[Y] + \frac{\text{Cov}(Y,X_1)}{\text{Var}(X_1)}X_1 + \frac{\text{Cov}(Y,X_2)}{\text{Var}(X_2)}X_2$.

   (b) Express the MSE for the estimator found in part (a) in terms of the variances of X_1, X_2, and Y and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.
   Solution: Substituting the values of b and c found into (1) yields

$$\text{MSE} = \text{Var}(Y) - \frac{\text{Cov}(Y, X_1)^2}{\text{Var}(X_1)} - \frac{\text{Cov}(Y, X_2)^2}{\text{Var}(X_2)}.$$

6. **[An estimation problem]**
Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u, v) = \begin{cases} \frac{8uv}{(15)^4} & u \geq 0, v \geq 0, u^2 + v^2 \leq (15)^2 \\ 0 & \text{else} \end{cases}$$

   (a) Find the constant estimator, δ^*, of Y, with the smallest mean square error (MSE), and find the MSE.
   Solution: We know $\delta^* = E[Y]$, and the resulting MSE is $\text{Var}(Y)$. We could directly compute the first and second moments of Y, but it is about the same amount work if f_Y is found first, so we find f_Y. The support of f_Y is [0,15]. For $0 \leq v \leq 15$,

$$f_Y(v) = \int_0^{\sqrt{225-v^2}} \frac{8uv}{15^4} du = \left. \frac{4u^2 v}{15^4} \right|_{u=0}^{\sqrt{225-v^2}} = \frac{4v}{225}\left(1 - \frac{v^2}{225}\right)$$

   Thus,

$$\delta^* = E[Y] = \int_0^{15} \frac{4v^2}{225}\left(1 - \frac{v^2}{225}\right) dv = 8,$$

   and

$$E[Y^2] = \int_0^{15} \frac{4v^3}{225}\left(1 - \frac{v^2}{225}\right) dv = 75,$$

   so MSE(using δ^*)=$\text{Var}(Y) = 75 - 8^2 = 11$.

(b) Find the unconstrained estimator, $g^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE.

Solution: We know $g^*(u) = E[Y|X=u]$. To compute g^* we thus need to find $f_{Y|X}(v|u)$. By symmetry, X and Y have the same distribution, so

$$f_X(u) = f_Y(u) = \begin{cases} \frac{4u}{225}\left(1 - \frac{u^2}{225}\right) & u \geq 0 \\ 0 & \text{else.} \end{cases}$$

Thus, $f_{Y|X}(v|u)$ is well defined for $0 \leq u \leq 15$. For such u,

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \begin{cases} \frac{2v}{225-u^2} & 0 \leq v \leq \sqrt{225-u^2} \\ 0 & \text{else.} \end{cases}$$

That is, for u fixed, the conditional pdf of Y has a triangular shape over the interval $\left[0, \sqrt{225-u^2}\right]$. Thus, for $0 \leq u \leq 15$,

$$g^*(u) = \int_0^{\sqrt{225-u^2}} \frac{2v^2}{225-u^2} dv = \frac{2\sqrt{225-u^2}}{3}.$$

To compute the MSE for g^* we find

$$E[g^*(X)^2] = \int_0^{15} g^*(u)^2 f_X(u) du = \int_0^{15} \frac{4(225-u^2)}{9} \frac{4u}{225}\left(1 - \frac{u^2}{225}\right) du = \frac{200}{3}.$$

Therefore, MSE(using g^*) $= E[Y^2] - E[g^*(X)^2] = \frac{25}{3} = 8.333\ldots$.

(c) Find the linear estimator, $L^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE. (Hint: You may use the fact $E[XY] = \frac{75\pi}{4} \approx 58.904$, which can be derived using integration in polar coordinates.)

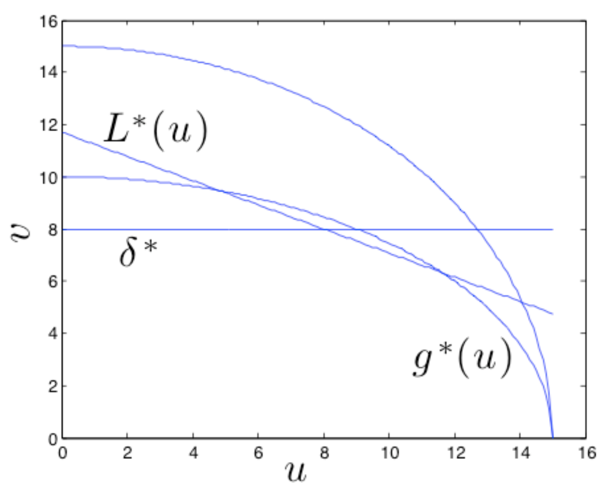
Solution: Using the hint, $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{75\pi}{4} - 64 \approx -5.0951$. Thus,

$$L^*(u) = E[Y] + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(u - E[X]) = 8 - (0.4632)(u-8)$$

and

$$\text{MSE(using } L^*) = \text{Var}(Y) - \frac{\text{Cov}(X,Y)^2}{\text{Var}(X)} = 8.6400$$

The three estimators are shown in the plot:



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