University of Illinois **Fall 2012**

ECE 313: Problem Set 1 version 2: Problems and Solutions
Axioms of probability and calculating the sizes of sets

Due: Wednesday, September 5 at 4 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Wednesdays, and is due by 4:00 p.m. on the following Wednesday. You must drop your homework into the ECE 313 lockbox (#20) in the Everitt Laboratory basement (east side) by this deadline. Late homeworks will not be accepted without prior permission from the instructor. All assignments must be submitted stapled if they consist of more than one sheet with the following heading in block capital letters with at least 12pt size font or equivalent neat handwriting, on the top right corner of the first page:
NAME AS IT APPEARS ON COMPASS
NETID
SECTION
PROBLEM SET #
The section should be one of X,C,D,F, and the problem set number an integer. Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. **[Moderating a discussion board]**
A moderator in an online discussion board tags comments based on two criteria: whether the post was made anonymously or not, and whether the comment is relevant or irrelevant. Consider the experiment of tagging of a comment.

(a) Define a sample space Ω describing the possible outcomes of this experiment. Explain how the elements of your set correspond to outcomes of the experiment.
Solution: One possible choice is $\Omega = \{(a,r) : a, r \in \{0,1\}\}$, or equivalently, $\Omega = \{(0,0),(0,1),(1,0),(1,1)\}$, where for a given outcome (a,r), a denotes whether the comment was made anonymously (1) or not (0), and r denotes whether the comment is relevant (1) or not (0).

(b) What is the cardinality of Ω?
Solution: $(2)(2) = 4$, because there are 2 possible choices for a, and 2 possible choices for r.

(c) Let A be the event that the comment is relevant. Specify the outcomes in A.
Solution: For Ω as defined in part (a), $A = \{\omega \in \Omega : r = 1\} = \{(0,1),(1,1)\}$.

(d) Let B be the event that the comment was not made anonymously. Specify the outcomes in B.
Solution: For Ω as defined in part (a), $B = \{\omega \in \Omega : a = 0\} = \{(0,0),(0,1)\}$.

(e) Specify the outcomes in $A^c \cup B$.
Solution: For Ω as defined in part (a), $A^c \cup B = \{(0,0),(1,0)\} \cup \{(0,0),(0,1)\} = \{(0,0),(0,1),(1,0)\}$

One could also use DeMorgan's law to obtain $A^c \cup B = (AB^c)^c = \{\omega \in \Omega : (a, r) \neq (1, 1)\}$.

(f) Assuming all outcomes are equally likely, find $P(A^c \cup B)$.

Solution: Following part (e), $P(A^c \cup B) = 1 - P(AB^c) = 1 - P\{(1, 1)\} = 3/4$.

2. **[A Karnaugh puzzle]**

Suppose $A, B,$ and C are events such that: $P(A) = P(B) = P(C) = 0.5$, $P(AB) = 0.3$, $P(B \cup C^c) = 0.55$, $P(A \cup C^c) = 0.7$, and $P(ABC^c) = 2P(A^cBC^c)$. Sketch a three event Karnaugh map showing the probabilities of each one of the map sections. Show your work.

Solution: One way to solve this is by letting $P(A^cBC^c) = x$, so that $P(ABC^c) = 2P(A^cBC^c) = 2x$. From the fact that $0.55 = P(B \cup C^c) = P(B) + P(C^c) - P(BC^c) = 0.5 + 0.5 - P(BC^c)$, one obtains $P(BC^c) = 0.45$. Then, because $P(BC^c) = P(ABC^c) + P(A^cBC^c) = x + 2x$, we get $x = 0.15$.

B^c		B		
			$x = 0.15$	A^c
			$2x = 0.3$	A
C^c	C		C^c	

Now, let $P(ABC) = y$, then, using the fact that $0.3 = P(AB) = P(ABC) + P(ABC^c) = y + 0.3$, one obtains $y = 0$.

B^c		B		
			0.15	A^c
		$y = 0$	0.3	A
C^c	C		C^c	

Let $P(A^cBC) = z$, then, from $0.5 = P(B) = P(ABC) + P(ABC^c) + P(A^cBC) + P(A^cBC^c) = 0 + 0.3 + z + 0.15$ one obtains $z = 0.05$.

B^c		B		
		$z = 0.05$	0.15	A^c
		0	0.3	A
C^c	C		C^c	

Let $P(AB^cC^c) = w$, then $0.7 = P(A \cup C^c) = P(A) + P(C^c) - P(AC^c) = 0.5 + 0.5 - P(AC^c)$, so that $P(AC^c) = 0.3$. Then, because $P(AC^c) = P(ABC^c) + P(AB^cC^c) = 0.3 + w$, one obtains $w = 0$.

B^c		B		
		$z = 0.05$	0.15	A^c
$w = 0$		0	0.3	A
C^c	C		C^c	

Let $P(AB^cC) = s$, then $0.5 = P(A) = P(ABC) + P(ABC^c) + P(AB^cC) + P(AB^cC^c) = 0 + 0.3 + s + 0$, and therefore $s = 0.2$.

B^c		B		
		$z = 0.05$	0.15	A^c
$w = 0$	$s = 0.2$	0	0.3	A
C^c	C		C^c	

Let $P(A^cB^cC) = t$, then $0.5 = P(C) = P(ABC) + P(AB^cC) + P(A^cBC) + P(A^cB^cC) = 0 + 0.2 + 0.05 + t$, so that $t = 0.25$.

B^c		B		
	$t = 0.25$	$z = 0.05$	0.15	A^c
$w = 0$	$s = 0.2$	0	0.3	A
C^c	C		C^c	

The remaining map section is then 0.05 because the sum of all probabilities has to add up to one. The completed map shown below.

B^c		B		
0.05	0.25	0.05	0.15	A^c
0	0.2	0	0.3	A
C^c	C		C^c	

3. **[Setting up teams]**
Suppose there are fifteen basketball players at a party, three of which are point-guards. The players are to be divided into three teams (T_1, T_2, and T_3) of five players, with all divisions being equally likely. Suppose that the labels of teams matter, so, for example, swapping all the players in one team for all the players in another team would be considered to give a different outcome.

   (a) Define a sample space Ω for this experiment.
   Solution: One choice is

$$\Omega = \{(T_1, T_2, T_3) : T_i \subset \{p_1, p_2, \ldots, p_{15}\}, |T_i| = 5, T_iT_j = \emptyset \text{ for all } i \neq j\}$$

   where p_i denotes the $i-th$ player.

   (b) Determine $|\Omega|$, the cardinality of Ω.
   Solution: $|\Omega| = \binom{15}{5}\binom{10}{5}\binom{5}{5} = \frac{15!}{(5!)^3} = 756,756$.
   There are at least two ways of doing this. One way is to choose the 5 players for T_1 from all 15 players, then choose the 5 players for team T_2 from the remaining 10 players, and put the remaining players in team T_3.
   Another way is order all 15 players (in 15! ways), assign the first five of them to team T_1, the second five to team T_2, and the remaining to team T_3. The order within teams is irrelevant, so we must remove the over counting by dividing by 5! to remove the ordering within T_1, also dividing by 5! to remove the ordering within T_2, and dividing by 5! to remove the ordering within T_3.

   (c) Let A be the event in which each one of the three teams has a point-guard. Find the cardinality of A.
   Solution: $=|A| = 3!\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{12!3!}{(4!)^3} = 207,900$.
   There are again at least two ways of doing this. One way is to assign one point-guard to each of the three teams (in 3! ways), then choose the remaining 4 players for T_1 from the 12 non-point-guard players. Once this is done, choose the remaining 4 players for team T_2 from the remaining 8 non-point-guard players, and put the remaining players in team T_3.
   Another way is order the 12 non-point-guard players (in 12! ways), assign the first four of them to team T_1, the second four to team T_2, and the remaining to team T_3. The order within teams is irrelevant, so we must remove the over counting by dividing by 4! to remove the ordering within T_1, also dividing by 4! to remove the ordering within T_2, and dividing by 4! to remove the ordering within T_3. Finally, assign a point-guard to each team, which can be done in 3! ways.

   (d) Find $P(A)$.
   Solution: $P(A) = \frac{|A|}{|\Omega|} = \frac{207,900}{756,756} \approx 0.2747$.

(e) Suppose now that the teams have no labels, so outcomes (T_1, T_2, T_3) and (T_2, T_1, T_3) would be indistinguishable for fixed T_1, T_2, T_3. Find $P(A)$.

Solution: $P(A) = \frac{207{,}900}{756{,}756} \approx 0.2747$, because both $|\Omega|$ and $|A|$ would be divided by 3! to get rid of the ordering of the teams.

4. **[Playing with cards]**

Suppose five cards are drawn from a standard 52 card deck of playing cards, as described in Example 1.4.3, with all possibilities being equally likely.

(a) *STRAIGHT* is the event that the five drawn cards have consecutive values (notice that the ace can be used as lower than 2 and also as higher than K). For all of you poker players, a straight flush is to also be considered a straight. Find $P(STRAIGHT)$.

Solution: The cardinality of the sample space is $\binom{52}{5}$. Suppose the consecutive values are $A, 2, 3, 4, 5$, then each one of these values can have one of four suits, so there are 4^5 ways of choosing such a consecutive sequence. The lowest value in a consecutive sequence can be any of $A, 2, 3, \ldots, 10$, so there are a total of $10(4^5)$ possible STRAIGHT possibilities. Thus,

$$P(STRAIGHT) \quad = \quad \frac{10(4^5)}{\binom{52}{5}} = \frac{10240}{2598960} \approx 0.0039$$

(b) Now, suppose that a wild card is added to the deck (the wild card is allowed to represent other existing cards). Find $P(STRAIGHT)$.

Solution: The cardinality of the sample space is now $\binom{53}{5}$.

If none of the cards drawn is a wild card, then the procedure from part (a) can be applied, and there are $10(4^5)$ possibilities.

If there is a wild card, then things change a bit. We'll distinguish between an 'outside' straight and an 'inside' straight. An outside straight is one where the four regular (non-wild card) cards are in consecutive order, so the wild card can be at the beginning or at the end of the sequence. The lowest ranked card in the four regular cards can be any of the 11 ranks $A, 2, 3, \ldots, 10, J$. Notice that, for example, if the four regular cards are $2, 3, 4, 5$ the wild card could act either as an A at the beginning of the sequence, or as a 6 at the end of the sequence, but as far as the cards that you have go, they are the same cards so both of these sequences should really be counted only once. Each one of the regular cards can have one of four suits, so there are $(11)4^4$ ways of choosing such outside straights because only four of the five cards have suits now (the wild card doesn't) Now, an inside straight is one where the wild card (W) is neither at the beginning nor at the end of the sequence, it can be the second, third, or fourth card in the sequence. Suppose the sequence is $A, W, 3, 4, 5$, then each one of the values $A, 3, 4, 5$ can have one of four suits, so there are 4^4 ways of choosing such a consecutive sequence. The lowest value in a consecutive sequence with the wild card in the second position can be any of $A, 2, \ldots, 10$, so there are a total of $10(4^4)$ possible straights with the wild card in the second position. The same reasoning applies when the wild card is either the third or the fourth card in the sequence, so there are $(3)10(4^4)$ possible inside straights. Thus,

$$P(STRAIGHT) \quad = \quad \frac{10(4^5) + 11(4^4) + (3)10(4^4)}{\binom{53}{5}} = \frac{20736}{2869685} \approx 0.0072$$

Notice that having a wild card increases the probability of $STRAIGHT$ by about 85%.

5. **[Working with probability axioms]**
Consider two events A and B.

   (a) Show that the probability that exactly one of the events A or B occurs equals $P(A) + P(B) - 2P(AB)$.

   Solution: Let E be the event that exactly one of the events A or B occurs, then E is the union of the two mutually exclusive events $E = AB^c \cup A^cB$. Therefore, $P(E) = P(AB^c \cup A^cB) = P(AB^c) + P(A^cB)$. Adding $P(AB)$ twice and subtracting it twice yields $P(E) = P(AB^c) + P(AB) - P(AB) + P(A^cB) + P(AB) - P(AB) = (P(AB^c) + P(AB)) - P(AB) + (P(A^cB) + P(AB)) - P(AB) = P(A) + P(B) - 2P(AB)$, where the last equality follows because A is the union of the mutually exclusive events $A = AB^c \cup AB$, and similarly for B.

   (b) Show that $P(AB^c) = P(A) - P(AB)$.

   Solution: $P(A) = P(A\Omega) = P(A(B \cup B^c)) = P(AB \cup AB^c) = P(AB) + P(AB^c)$, where the last equality follows from the fact that B and B^c are mutually exclusive. Solving for $P(AB^c)$ gives the desired result.

   (c) Show that $P(A^cB^c) = 1 - P(A) - P(B) + P(AB)$.

   Solution: $1 = P(\Omega) = P(A \cup A^c) = P(A \cup A^c(\Omega)) = P(A \cup A^c(B \cup B^c)) = P(A \cup A^cB \cup A^cB^c) = P(A) + P(A^cB) + P(A^cB^c)$, where the last equality follows from the fact that the three events are mutually exclusive. Solving for $P(A^cB^c)$ gives $P(A^cB^c) = 1 - P(A) - P(A^cB)$. Adding and subtracting $P(AB)$ yields $P(A^cB^c) = 1 - P(A) - P(A^cB) - P(AB) + P(AB) = 1 - P(A) - (P(A^cB) + P(AB)) + P(AB) = 1 - P(A) - P(B) + P(AB)$, where the last equality follows because A^cB and AB are mutually exclusive events such that $A^cB \cup AB = B$.