ECE 313: Problem Set 12
Moments of jointly distributed random variables, minimum mean square error estimation

Due: Wednesday December 5 at 4 p.m.
Reading: *313 Course Notes* Sections 4.8-4.9

1. **[Covariance I]**
   Consider random variables X and Y on the same probability space.
   (a) If $\text{Var}(X + 2Y) = 40$ and $\text{Var}(X - 2Y) = 20$, what is $\text{Cov}(X, Y)$?
   (b) In part (a), determine $\rho_{X,Y}$ if $\text{Var}(X) = 2 \cdot \text{Var}(Y)$.

2. **[Covariance II]**
   Suppose X and Y are random variables on some probability space.
   (a) If $\text{Var}(X + 2Y) = \text{Var}(X - 2Y)$, are X and Y uncorrelated ?
   (b) If $\text{Var}(X) = \text{Var}(Y)$, are X and Y uncorrelated?

3. **[Covariance III]**
   Rewrite the expressions below in terms of $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$, and $\text{Cov}(X, Y)$.
   (a) $\text{Cov}(3X + 2, 5Y - 1)$
   (b) $\text{Cov}(2X + 1, X + 5Y - 1)$.
   (c) $\text{Cov}(2X + 3Z, Y + 2Z)$ where Z is uncorrelated to both X and Y.

4. **[Covariance IV]**
   Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The *signal-to-noise-ratio* (SNR) of the observation X_1 or X_2 is defined as the ratio $SNR_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.
   (a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.
   (b) The system designer notices that the averaging strategy is giving $SNR_S = (1.5)SNR_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient $\rho_{X_1X_2}$.
   (c) Under what condition on $\rho_{X,Y}$ can the averaging strategy result in an SNR_S that is arbitrarily high?

5. **[Linear minimum MSE estimation from uncorrelated observations]**
   Suppose Y is estimated by a linear estimator, $L(X_1, X_2) = a + bX_1 + cX_2$, such that X_1 and X_2 have mean zero and are uncorrelated with each other.
   (a) Determine a, b and c to minimize the MSE, $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of $E[Y]$, the variances of X_1 and X_2, and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

(b) Express the MSE for the estimator found in part (a) in terms of the variances of X_1, X_2, and Y and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

6. **[An estimation problem]**
Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{8uv}{(15)^4} & u \geq 0, v \geq 0, u^2 + v^2 \leq (15)^2 \\ 0 & \text{else} \end{cases}$$

(a) Find the constant estimator, δ^*, of Y, with the smallest mean square error (MSE), and find the MSE.

(b) Find the unconstrained estimator, $g^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE.

(c) Find the linear estimator, $L^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE. (Hint: You may use the fact $E[XY] = \frac{75\pi}{4} \approx 58.904$, which can be derived using integration in polar coordinates.)