ECE 313: Problem Set 11
Working with joint densities

Due: Wednesday November 28 at 4 p.m.
Reading: *313 Course Notes* Sections 4.5-4.7

1. **[Sum of two independent geometric random variables]**
Let X be a geometric random variable with parameter p, and Y be a geometric random variable with parameter q, with X and Y being independent. Let $S = X + Y$.

   (a) Suppose $0 < q < p < 1$. Find the pmf of S using convolution. Simplify your expression as much as possible (in particular, don't leave the answer as a summation.)

   (b) Now consider the special case of $p = q$. Find the pmf of S using convolution.

   (c) Verify your answer to part (b) using the negative binomial distribution.

2. **[Sum of two independent continuous-type random variables]**
Suppose X and Y have the joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 2v & \text{if } 0 \le u \le 1,\ 0 \le v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let $S = X + Y$. Find the pdf of S, i.e., find $f_S(s)$.

   Hint: You may want to consider the cases $s < 0$, $0 \le s < 1$, $1 \le s \le 2$, and $s > 2$ separately.

3. **[Sum of two dependent continuous-type random variables]**
We know from PS 10 that the random variables X and Y with the following joint pdf are dependent.

$$f_{X,Y}(u,v) = \begin{cases} 6e^{-(u+2v)} & \text{if } 0 \le u < v \\ 0 & \text{otherwise} \end{cases}$$

   Find the pdf of the sum $S = X + Y$,

4. **[Rayleigh fading and selection diversity]**
The signal strength received at a base-station from a cellular phone in a dense urban environment can be modeled quite well by a Rayleigh random variable. When a cellular phone user is around the midpoint between two base-stations, some systems use *selection diversity*, where the base station with the larger received signal strength is used to decode the user's message. Let X and Y denote the signal strengths received at the two base-stations. Then

$$f_X(u) = f_Y(u) = \begin{cases} ue^{-\frac{u^2}{2}} & \text{if } u \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

   Assume that X and Y are independent.

   (a) Find the mean signal strength at each base-station, i.e., find $E[X]$.
   Hint: Convert the integral required to compute $E[X]$ into one that corresponds to finding the variance of a $N(0,1)$ random variable.

   (b) Find the pdf of the signal strength chosen by the selection diversity system, i.e., find the pdf of $Z = \max(X,Y)$.

   (c) Find the mean signal strength after selection diversity, i.e., find $E[Z]$.

5. **[Joint pdf of functions of random variables]**
Let X and Y be random variables, and define two other random variables as $W = \frac{1}{\sqrt{2}}(X - Y)$ and $Z = \frac{1}{\sqrt{2}}(X + Y)$.

   (a) If X and Y are *independent* $N(0, 1)$ random variables, then show that W and Z are also *independent* $N(0, 1)$ random variables.

   (b) Now consider the case where X and Y are still independent, but $X \sim N(0, 1)$ whereas $Y \sim N(0, 2)$. Find the joint distribution of W and Z. Are W and Z independent?