University of Illinois **Fall 2012**

ECE 313: Problem Set 6
Reliability and cumulative distribution functions (CDFs)

Due: Wednesday, October 10 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.12 & 3.1
Reminder: Exam 1 is on Monday, October 8, 7-8:15 p.m.

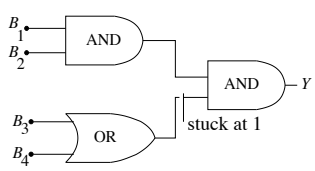
1. **[The reliability of a hierarchical backup system]**
Consider a parallel storage system composed of nine subsystems, each of which contains nine servers. Each subsystem can tolerate a single server failure, and the overall system can tolerate a single subsystem failure. Thus, in order for the overall system to fail, there has to be at least two subsystems that each have at least two server failures. Suppose servers fail independently with probability p.

   (a) Find an expression for the exact probability, p_0, that a particular subsystem fails, in terms of p. Also, compute the numerical value of p_0 assuming that $p = 0.001$.

   (b) Find an expression for the exact probability, p_1, that the overall system fails, in terms of p_0. Also, compute the numerical value of p_1 assuming that $p = 0.001$.

   (c) Give an upper bound on p_0 and an upper bound on p_1 using the union bound, and compute their numerical values assuming that $p = 0.001$.

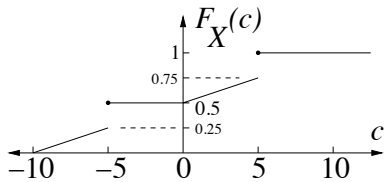
2. **[Fault detection in a Boolean circuit]**
Consider the Boolean circuit shown, with two AND gates, one OR gate, four binary input variables $B_1, \ldots, B_4$, and binary output variable Y.



   (a) Suppose there is a *stuck at one* fault as shown, so that the value one is always fed into the second AND gate, instead of the output of the OR gate. Assuming that $B_1, \ldots, B_4$ are independent and equally likely to be zero or one, what is the probability that the output value Y is incorrect?

   (b) Suppose that the circuit is working correctly with probability 0.5, or has the indicated stuck at one fault with probability 0.5. Suppose three distinct randomly generated test patterns are applied to the circuit. (Here, a test pattern is a binary sequence of length four. Assume all sets of three distinct test patterns are equally likely.) Given that the output is correct on all three of the patterns, what is the conditional probability the circuit is faulty?

3. **[Using a CDF]**
The CDF of a random variable X is shown below. Find the numerical values of the following.



(a) $P\{X = 5\}$

(b) $P\{X = 0\}$

(c) $P\{|X| \leq 5\}$

(d) $P\{X^2 \leq 4\}$

4. **[A continuous approximation of the Zipf distribution]**
Let M be a positive integer and let $\alpha > 0$. Let Y be a random variable with the pdf

$$f_Y(u) = \begin{cases} \frac{u^{-\alpha}}{C} & 0.5 \leq u \leq M + 0.5 \\ 0 & \text{else.} \end{cases}$$

(Note that for integer values of u, the pdf f_Y is the same as the Zipf pmf p for X appearing in problem set 2, and for $1 \leq k \leq M$, $p(k) \approx \int_{k-0.5}^{k+0.5} f_Y(u)du$. An advantage of using f_Y is that it can be analytically integrated, whereas there is no closed form expression for the CDF of the Zipf pmf.)

(a) Express the constant C in terms of M and α. (For simplicity, assume $\alpha \neq 1$.)

(b) If $M = 2000$ and $\alpha = 0.8$, what is $P\{Y \leq 500.5\}$? (This approximates $P\{X \leq 500\}$ from problem set 2.)