University of Illinois **Fall 2012**

ECE 313: Problem Set 3
Conditional probabilities, independence, and the binomial distribution

Due: Wednesday September 19 at 4 p.m.
Reading: *313 Course Notes* Sections 2.3–2.4

1. [**To diet or not to diet**]
Cookie Monster wants to eat a cookie but needs to go on a diet. So he decides to toss a fair coin to decide whether or not to eat the cookie. He will eat the cookie if the toss is heads. If the toss is tails, he does not give up immediately, but decides to flip the coin two more times. If the subsequent two tosses are both heads, then he eats the cookie. If not, he is good and refrains from eating the cookie.
   (a) Find the probability that Cookie Monster eats the cookie.
   (b) What is the probability that the first toss is heads, given that Cookie Monster eats the cookie.

2. [**Biased coins**]
You are given two biased coins that look identical, but one of them is biased to show heads with probability 1/3 and the other is biased to show heads with probability 2/3. You are not told which coin is which. You flip each coin once, choosing the first coin to flip at random (i.e., each coin has probability one half of being chosen as the first). Define the events H_i = "i-th toss is heads" , $i = 1, 2$.
   (a) Find $P(H_1)$ and $P(H_2)$?
   (b) Find the probability that both tosses are heads, i.e., find $P(H_1H_2)$?
   (c) Are H_1 and H_2 independent? Justify your answer.

3. [**Independence**]
This problem tests your understanding of independence. You may find part (a) to be useful in answering parts (b) or (c).
   (a) Prove the following statement:
   If A and B are independent and $B \subset A$, then either $P(B) = 0$ or $P(A) = 1$.
   (b) Consider tossing a fair coin n times. Let $n = 3$. Are events A = "two or more tails" and B ="one or two heads" independent? Justify your answer.
   (c) Repeat part (b) for the case $n = 4$.

4. [**Cell phone call initiation**]
When you press "CALL" on your cell phone, the phone sends a "Call Initiate Signal (CIS)" to nearest base station. The phone waits for a response for some pre-specified amount of time. If a positive response is received in that time, the call is initiated. If no response is received, it sends the CIS again. If it does not get a response after n tries, it stops sending the CIS and issues a busy signal. Assume all the CIS transmissions are independent, and the probability that a CIS transmission gets a positive response is q. Let X denote the random variable that represents the number of times a CIS is sent during a call attempt.
   (a) Find the pmf of X.
   (b) Find the probability that you will get a busy signal.
   (c) If $q = 0.9$, and your cell phone provider want to make sure that the probability that you will get a busy signal is less than or equal to 0.001. What is the smallest value of n that the cell phone provider can use?

5. **[Error probability in digital communication]**
Consider a communication system in which a byte (8 bits) is transmitted over a channel that introduces bit errors independently across the byte, with each bit error having probability 0.2. Let Y denote the number of bit errors introduced by the channel.

   (a) Find the pmf of Y and $E[Y]$.

   (b) What is the most probable number of errors in the received byte?

   (c) Given that at least one error has occurred, find the probability that exactly two errors have occurred.

   (d) Now suppose the receiver can use a code to correct up to two errors in the received byte. Find the probability that the byte is decoded correctly.

6. **[Foopon]**
Foopon, a company that offers daily coupon deals, has the following business arrangement with a restaurant. A coupon for a \$40 meal can be purchased by a consumer for \$20 from Foopon, and the restaurant and Foopon split the \$20 equally. Foopon sends an email message to a list of 100 potential consumers, who act independently, each of whom decides to buy a coupon with probability 0.1. The deal is "on" if at least 10 consumers decide to purchase a coupon. If only 9 or fewer consumers decide to purchase a coupon, then the deal is "off" and no coupons are sold.

   (a) What is the probability that the deal is "on"?

   Hint: When you compute summations involving binomial coefficients on a computer, you may run into numerical precision issues when n is large and p is small as it is in this problem. To avoid these numerical issues, you may want to compute the probability that the deal is *not* "on" first.

   (b) Given that the deal is "on", what is the probability that exactly 15 coupons are sold.

   (c) Calculate the expected profit made by Foopon from this arrangement.

   Hint: Here again, to avoid numerical precision issues, you may want to compute this expectation indirectly, by exploiting the fact that the expected value of a binomial with parameters n and p is equal to np.

   (d) If it costs \$5 for the restaurant to prepare each meal, then calculate the expected profit made by restaurant through this arrangement with Foopon.

   (e) The restaurant calculates that if it did not make the deal with Foopon, the same 100 consumers, acting independently, would choose to eat at the restaurant and pay the full \$40, each with probability 0.02 (i.e., using Foopon increases by five-fold the probability that a consumer would choose the restaurant). Assuming (as before) that it costs \$5 for the restaurant to prepare each meal, calculate the expected profit that the restaurant would make without Foopon.