**University of Illinois** Fall 2012

# ECE 313: Final Exam

Monday, December 17, 2012, 8:00 - 11:00 a.m.
151 Everitt Lab (Sections X and C) , 165 Everitt Lab (Section D), & 269 Everitt Lab (Section E)

1. (a) The random variable $X_3 + \cdots + X_n$ is independent of $(X_1, X_2)$ and has the binomial distribution with parameters $n-2$ and $p$.

$$\begin{aligned} P(S = 2|X_1 = 0, X_2 = 1) &= P(X_3 + \cdots + X_n = 1|X_1 = 0, X_2 = 1) \\ &= P\{X_3 + \cdots + X_n = 1\} \\ &= \binom{n-2}{1} p(1-p)^{n-3} \\ &= (n-2)p(1-p)^{n-3}. \end{aligned}$$

(b) The random variable $S$ has the binomial distribution with parameters $n$ and $p$, and $X_3 + \cdots + X_n$ has the binomial distribution with parameters $n-2$ and $p$. Hence,

$$\begin{aligned} P(X_1 = 0, X_2 = 1|S = 2) &= \frac{P\{X_1 = 0, X_2 = 1, S = 2\}}{P\{S = 2\}} \\ &= \frac{P\{X_1 = 0, X_2 = 1, X_3 + \cdots + X_n = 1\}}{P\{S = 2\}} \\ &= \frac{(1-p)p\binom{n-2}{1}p(1-p)^{n-3}}{\binom{n}{2}p^2(1-p)^{n-2}} = \frac{2(n-2)}{n(n-1)} \end{aligned}$$

2. (a) Let $s^*$ be the highest rate server. There are $\binom{7}{2}$ ways to select three servers including $s^*$, and $\binom{8}{3}$ ways to select three of the eight servers, so $P(A) = \frac{\binom{7}{2}}{\binom{8}{3}} = 3/8$.

ALTERNATIVELY, Let $s^*$ be the highest rate server. The probability $s^*$ is not sampled first is 7/8. Given $s^*$ is not sampled first, it is not sampled second with probability 6/7. Given $s^*$ is not sampled first or second, it is not sampled third with probability 5/8. Hence, $P(A) = 1 - (7/8)(6/7)(5/6) = 3/8$.

ALTERNATIVELY: $P(A) = 1 - \frac{\binom{7}{3}}{\binom{8}{3}} = 1 - \frac{7\cdot 6\cdot 5}{8\cdot 7\cdot 6} = 3/8$.

ALTERNATIVELY: A way to think about this is that the rates are assigned to the servers after the selection of three servers is made. The probability the highest rate is assigned to one of the three sampled servers is 3/8. So $P(A) = 3/8$.

(b) Let $S$ denote the set of four slowest servers. $B$ is the event that all three servers sampled are in $S$. The probability the first one sampled is in $S$ is 4/8. Given the first one sampled is in $S$, the probability the second one sampled is in $S$ is 3/7. Given the first two sampled are in $S$, the third one sampled is in $S$ with probability 2/6. So $P(B) = (4/8)(3/7)(2/6) = \frac{1}{14}$.

ALTERNATIVELY: There are $\binom{8}{3}$ ways to choose which servers to sample and $\binom{4}{3}$ ways to select the three servers to sample from among the four servers with the lowest service rates. So $P(B) = \frac{\binom{4}{3}}{\binom{8}{3}} = \frac{4\cdot 3!}{8\cdot 7\cdot 6} = \frac{4}{56} = \frac{1}{14}$.

3. (a) Since counts cannot be dropped when $X = 0$

$$P(Y = k|X = 0) = \begin{cases} 1-\epsilon & \text{if } k = 0 \\ \epsilon & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$P(Y = k|X = \ell) = \begin{cases} \epsilon & \text{if } k = \ell - 1 \\ 1-2\epsilon & \text{if } k = \ell \\ \epsilon & \text{if } k = \ell + 1 \\ 0 & \text{else} \end{cases}$$

(c) In order for $Y = 1$, is must be that $X \in \{0, 1, 2\}$. So

$$\begin{aligned} P\{Y = 1\} &= P(Y = 1|X = 0)P\{X = 0\} + P(Y = 1|X = 1)P\{X = 1\} + P(Y = 1|X = 2)P\{X = 2\} \\ &= \epsilon e^{-\lambda} + (1-2\epsilon)\lambda e^{-\lambda} + \epsilon\frac{\lambda^2 e^{-\lambda}}{2}. \end{aligned}$$

(d) By Bayes rule

$$\begin{aligned} P(X = 1|Y = 1) &= \frac{P(Y = 1|X = 1)P\{X = 1\}}{P\{Y = 1\}} \\ &= \frac{(1-2\epsilon)e^{-\lambda}\lambda}{\epsilon e^{-\lambda} + (1-2\epsilon)e^{-\lambda}\lambda + \epsilon e^{-\lambda}\lambda^2/2} \\ &= \frac{2\lambda(1-2\epsilon)}{(2+\lambda^2)\epsilon + 2\lambda(1-2\epsilon)} \end{aligned}$$

4. (a)

$$F_X(c) = \begin{cases} 0, & c < 0 \\ \int_0^c \frac{1}{2}(2-u)\ du = c - \frac{c^2}{4}, & 0 \le c < 2 \\ 1, & c \ge 2 \end{cases}$$

(b) $\mathrm{P}\{X > 1\} = 1 - \mathrm{P}\{X \le 1\} = 1 - F_X(1) = \frac{1}{4}$

(c) $\mathrm{P}\{X > 1 \mid X \le 2\} = \frac{P\{X>1, X\le 2\}}{P\{X \le 2\}} = \frac{F_X(2)-F_X(1)}{F_X(2)} = \frac{1-\frac{3}{4}}{1} = \frac{1}{4}$

5. For $0 \le c \le b$, $F_X(c) = \int_0^c f_X(u)du = \int_0^c a\cos(u)du = a(\sin(c) - \sin(0)) = a\sin(c)$.
In particular,

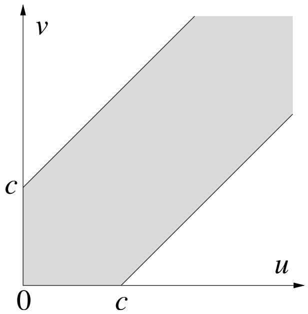
$$F_X(\pi/4) = a\sin(\pi/4) = a\frac{1}{\sqrt{2}}.$$

Using the given information, $F_X(\pi/4) = \frac{1}{\sqrt{2}}$, it follows that $a = 1$.
Thus, $F_X(c) = \sin(c)$ for $0 \le c \le b$. In order for $f_X(u)$ to be a valid pdf, it must be that $F_X(b) = 1$, or $\sin(b) = 1$. Solving for $b$ gives $b = \pi/2$.

Therefore, $F_X(c) = \begin{cases} 0 & c < 0 \\ \sin(c) & 0 \le c < \frac{\pi}{2} \\ 1 & c \ge \frac{\pi}{2} \end{cases}$

6. (a) Since $X$ and $Y$ can each take on any nonnegative values, the same is true of $S$. So the support of $f_S$ is the set of nonnegative numbers, $[0, \infty)$.

(b)



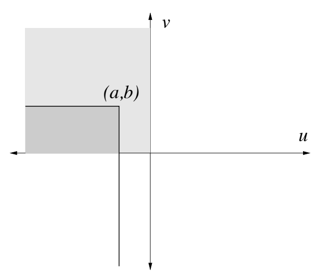

(c) One method of solution was mentioned in problem set 10 #4. Imagine $X$ and $Y$ are the failure times of two machines. Once one machine fails (it doesn't matter which one because they are identical, or when, because the lifetimes have the memoryless property) the remaining lifetime $S$ of the other machine has the exponential distribution with parameter one. That is, $f_S(c) = \begin{cases} e^{-c} & c \geq 0 \\ 0 & c < 0 \end{cases}$.

ALTERNATIVELY, the joint density is $e^{-u}e^{-v}$ over the region found in part $(b)$, and by symmetry the integral of the pdf over the region is twice the integral of the pdf over the half of the region above the $u = v$ line. Hence,

$$\begin{aligned} F_S(c) &= 2\int_0^\infty \int_u^{u+c} e^{-u}e^{-v} dv du = 2\int_0^\infty e^{-u}\left[-e^{-v}\right]\Big|_u^{u+c} du \\ &= 2\int_0^\infty e^{-u}\left(e^{-u} - e^{-(u+c)}\right) du \\ &= \left(1 - e^{-c}\right)\int_0^\infty 2e^{-2u} du = 1 - e^{-c}, \end{aligned}$$

which is the CDF of an exponential with parameter 1. That is, $f_S(c) = \begin{cases} e^{-c} & c \geq 0 \\ 0 & c < 0 \end{cases}$.

7. The support of the joint pdf is the upper left quadrant of the plane, so $F_{X,Y}(a, b)$ is the integral of $2e^{u-2v}$ over the intersection of the upper left quadrant and $\{(u, v) : u \leq a,\ v \leq b\}$, as illustrated in the figure.




For $b < 0$, $F_{X,Y}(a, b) = 0$ because the intersection is empty.

For $a < 0, b > 0$: $F_{X,Y}(a, b) = \int_{-\infty}^{a}\left[\int_0^b 2e^{u-2v} dv\right] du = e^a(1 - e^{-2b})$.

For $a > 0, b > 0$: $F_{X,Y}(a, b) = F_{X,Y}(0, b) = (1 - e^{-2b})$, because the intersection is the same as the intersection of the upper left quadrant and $\{(u, v) : u \leq 0,\ v \leq b\}$.

8. (a) $\text{Cov}(X+Y, X-Y) = \text{Cov(X,X)} - \text{Cov(X,Y)} + \text{Cov(Y,X)} - \text{Cov(Y,Y)} = \text{Var}(X) - \text{Var}(Y) = 0$.
   (b) $\text{Cov(3X+Z,3X+Y)} = 9\text{Var}(X) + 3\text{Cov}(X,Y) + 3\text{Cov}(Z,X) + \text{Cov}(Z,Y) = 9 \cdot 20 + 3 \cdot 10 + 3 \cdot 10 + 5 = 245$.
   (c) Since $E[X+Y] = 0$, $E[(X+Y)^2] = \text{Var}(X+Y) = \text{Var}(X) + 2\text{Cov}(X,Y) + \text{Var}(Y) = 20 + 2 \cdot 10 + 20 = 60$.
   (d) $\widehat{E}[Y+Z|X=3] = \frac{\text{Cov}(Y+Z,X) \cdot 3}{\text{Var}(X)} = \frac{(10+10) \cdot 3}{20} = 3$.

9. (a) First note that $\text{Cov}(X,Y) = \sigma_X \sigma_Y \rho_{XY} = 0.5$. Therefore

$$\text{Cov}(X,W) = \text{Cov}(X, X - \alpha Y) = \text{Var}(X) - \alpha \text{Cov}(X,Y) = 1 - 0.5\alpha,$$

   and $\text{Cov}(X,W) = 0$ if $\alpha = 2$. Since $X$ and $W$ are jointly Gaussian, $\alpha = 2$ also makes them independent.
   (b) By part (a) $X$ and $W$ are independent. Therefore, the unconstrained MMSE estimator, $E[X|W] = E[X] = 0$, and the corresponding MSE $= \text{Var}(X) = 1$.
   (c) $E[Z] = E[2X + Y + 2] = 2E[X] + E[Y] + 2 = 2$, and

$$\text{Var}(Z) = 4\text{Var}(X) + \text{Var}(Y) + 4\text{Cov}(X,Y) = 4 + 4 + 2 = 10$$

   (d) Since $Y$ and $Z$ are jointly Gaussian, the unconstrained MMSE estimator is the same as the linear MMSE estimator of $Y$ based on $Z$, which is given by

$$g^*(Z) = L^*(Z) = E[Y] + \frac{\text{Cov}(Y,Z)}{\text{Var}(Z)}(Z - E[Z])$$

   Now

$$\text{Cov}(Y,Z) = \text{Cov}(Y, 2X + Y + 2) = 2\text{Cov}(Y,X) + \text{Var}(Y) = 1 + 4 = 5$$

   and therefore

$$L^*(Z) = 0 + \frac{5}{10}(Z-2) = \frac{1}{2}(Z-2)$$

10. (a)

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^u uv(4u^2)dvdu \\ &= \int_0^1 4u^3 \left( \int_0^u vdv \right) du \\ &= \int_0^1 2u^5 du = \frac{1}{3} \end{aligned}$$

   (b)

$$f_Y(v) = \int_v^1 4u^2 \, du = \begin{cases} \frac{4}{3}(1 - v^3), & 0 \le v \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (c)

$$f_{X|Y}(u|v) = \begin{cases} 0, & 0 < v < 1, \quad 0 < u < v \\ \frac{4u^2}{\frac{4}{3}(1-v^3)} = \frac{3u^2}{1-v^3}, & 0 < v < 1, \quad v < u < 1 \\ \text{undefined}, & v < 0 \text{ or } v > 1 \end{cases}$$

   (d) For $0 < v < 1$, $E[X^2|Y=v] = \int_v^1 u^2 \, \frac{3u^2}{1-v^3} \, du = \frac{3}{5} \frac{1-v^5}{1-v^3}$

11. (a) False, False
    (b) True,False
    (c) True, True, False
    (d) True, False, True